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Scissors congruences and the bar and cobar constructions

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Abstract

A Hopf algebra of spherical polytopes introduced more than 20 years ago by C.H. Sah, in his book on scissors congruences, is revisited through the light of shuffle algebra. Oddly enough, these two topics where considered separately in the book of Sah, but not related. Their connection is based on previous work of Dupont. This new point of view is applied to the cohomology of Dehn complexes, considered recently by Goncharov. We get in particular a general procedure for linking this cohomology to the homology of some classical Lie groups, considered as discrete groups. A crucial role is played by certain bar and cobar constructions.

The late Chi-Han Sah wrote in his 1977 monograph (Sah, Hilbert's third problem: scissors congruence, Research Notes in Math 33, Pitman, 1979): "In my opinion, the third problem of Hilbert is of equal stature with the rest of the problems of Hilbert". The works of Dupont, Sah and others have amply justified this vision. The following article would like to yet reinforce it. One can look at the recent book of Dupont (Scissors Congruences, Group Homology and Characteristic Classes, Nankai Tracts in Mathematics, Vol. 1, World Scientific, Singapore, 2001) to get an idea of the richness and state of the problem.

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1. Introduction

1.1. Extended version of Hilbert's third problem

Currently, one can see Hilbert's third problem, on *scissors congruences* of polytopes in classical geometries Euclidean, spherical or hyperbolic, as composed of two different

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questions. The first concerns the *Dehn invariants* and the cohomology groups of *Dehn complexes*: these complexes have been recently introduced by Goncharov [6], but were already implicit in the book of Sah, and their cohomology is suspected to be related to the homology of classical Lie groups, considered as discrete groups. The second one concerns the effective calculation of these homology groups of classical groups: a better understanding of these, at least for low dimensions, relies on K-theory and on the consideration of invariants like volumes, Cheeger–Chern–Simons classes and regulators.

The dimension two is elementary. The program was completed in Euclidean geometry, in dimension three and four, by Sydler and Jessen. In spherical and hyperbolic geometry in dimension three, the first question has been settled by Dupont and Sah. Except for those special cases, very little is known and the problem is wide open.

1.2. Dehn complexes

The subject of the paper is the cohomology of Dehn complexes. We develop a machinery which connects this cohomology to the homology of isometry groups, considered as discrete groups, for the three classical geometries and in any odd dimension. To explain some of the results, let us limit ourselves in this introduction to spherical geometry. To each odd-dimensional sphere S^{2m-1} is associated, in Section 8, a scissors congruence group of polytopes $\bar{\mathscr{P}}(S^{2m-1})$ (the corresponding groups are 0 in even dimensions and by convention $\bar{\mathscr{P}}(S^{2m-1}) = \mathbb{Q}$, for $m = 0$).

After Sah [19, Chapter 6] the direct sum

$$\mathbf{P}(S) = \bigoplus_{m\geqslant 0} \bar{\mathscr{P}}(S^{2m-1}),$$

has a natural structure of graded-commutative Hopf algebra, where $\bar{\mathscr{P}}(S^{2m-1})$ is in degree $2m$. The product

$$\bar{\mathscr{P}}(S^{2m_1-1}) \otimes \bar{\mathscr{P}}(S^{2m_2-1}) \to \bar{\mathscr{P}}(S^{2(m_1+m_2)-1}),$$

is induced by the *join* of polytopes, and the reduced coproduct is given by the so-called *total Dehn invariant*, which generalizes the classical Dehn invariant in dimension 3

$$\mathscr{D} : \bar{\mathscr{P}}(S^{2m-1}) \to \bigoplus_{i=1}^{m-1} \bar{\mathscr{P}}(S^{2(m-i)-1}) \otimes \bar{\mathscr{P}}(S^{2i-1}).$$

We call this Hopf algebra the *Sah algebra*. For the grading induced on the cohomology of the Sah algebra $\mathbf{P}(S)$, the piece of degree $2m$ is the cohomology of the $2m$-part of the graded reduced cobar complex of $\mathbf{P}(S)$ as a coalgebra. This is precisely the mth Dehn complex. It looks like

$$\mathscr{D}(m) : \bar{\mathscr{P}}(S^{2m-1}) \to \bigoplus_{m_1+m_2=m} \bar{\mathscr{P}}(S^{2m_1-1}) \otimes \bar{\mathscr{P}}(S^{2m_2-1}) \to \cdots$$

$$\to \bigoplus_{m_1+\cdots+m_s=m} \bar{\mathscr{P}}(S^{2m_1-1}) \otimes \cdots \otimes \bar{\mathscr{P}}(S^{2m_s-1}) \xrightarrow{d} \cdots \xrightarrow{d} \otimes^m \bar{\mathscr{P}}(S^1),$$

with differential

$$d = \mathscr{D} \otimes id \otimes \cdots \otimes id - id \otimes \mathscr{D} \otimes id \otimes \cdots \otimes id + \cdots \pm id \otimes \cdots \otimes id \otimes \mathscr{D},$$

where $\bar{\mathscr{P}}(S^{2m-1})$ is in cohomological degree one, and the integers m_i are non-zero.

Let $H_*(O(n,\mathbb{R}),\mathbb{Q}^t)$ be the homology of the orthogonal group $O(n,\mathbb{R})$, considered as a discrete group, acting on $\mathbb{Q}$ by the determinant, one of our main results in Section 10 (Theorem 10.1.1, 10.1.2), extending a result of Dupont for $m=2$, is:

Theorem. *The cohomology of the Dehn complex $\mathscr{D}(m)$, in degrees m and $m-1$, is given by*

$$H^m(\mathscr{D}(m)) \cong H_m(O(2m,\mathbb{R}),\mathbb{Q}^t),$$

$$H^{m-1}(\mathscr{D}(m)) \cong H_{m+1}(O(2m,\mathbb{R}),\mathbb{Q}^t).$$

A related result is:

Theorem. *The natural map*

$$\bigotimes^m H_1(O(2,\mathbb{R}),\mathbb{Q}^t) \to H_m(O(2m,\mathbb{R}),\mathbb{Q}^t)$$

is surjective.

These two results present some analogy with the appearance of Milnor K-Theory in motivic cohomology [6].

There is also, in Section 10 (see Theorem 10.1.4), a precise description of $H^1(\mathscr{D}(3))$, as a quotient of the homology group $H_5(O(6,\mathbb{R}),\mathbb{Q}^t)$. This reduces the spherical Hilbert's third problem for S^5, up to torsion, to the study of the homology of $O(6,\mathbb{R})$, considered as a discrete group. We give also the counterparts of the previous results in hyperbolic or Euclidean geometry.

1.3. Strategy and structure of the paper

Our strategy is to revisit the Sah algebra from the point of view of *Tits buildings* and *Steinberg modules*, used by Dupont in [2]. One aspect of the paper of Dupont was to connect scissors congruence groups to Steinberg modules. Scissors congruence groups are more or less groups of coinvariants of Steinberg modules under actions of isometry groups: these are homology groups in degree 0. To put scissors congruence groups in a wider landscape, it is important to consider also the homology of isometry groups, with Steinberg modules coefficients, in higher degrees. A great deal of the paper is devoted to their study, of interest in itself. This explains why scissors congruence groups appear only in Section 8.

We begin in Section 2 by introducing a new point of view on Tits buildings [3] for classical geometries: the main simple, but crucial idea, is to replace a flag of subspaces in a Euclidean vector space by an orthogonal direct sums decomposition of this vector space. In particular this enables one to use symmetric groups actions.

As a first application we get in this section the existence of weights corresponding to decompositions of spherical Steinberg modules, under actions of orthogonal groups.

In Section 3 we assemble the homology groups of orthogonal groups, in all dimensions, to get an algebra called the orthogonal algebra, which plays a central role all over the paper.

In Section 4, the homologies of isometry groups, with coefficients in the determinant action on one hand and the Steinberg modules twisted by the determinant on the other, are connected by a spectral sequence through the orthogonal algebra. This uses the Salomon–Tits theorem. On the other hand, the bar construction over the orthogonal algebra pops up naturally in the study of this spectral sequence.

Section 5 applies the previous spectral sequence to perform some calculations used in what follows. This incorporates in particular previous results of Dupont.

It is well known that the tensor algebra over a vector space has a natural structure of commutative Hopf algebra, where the product is the shuffle product and the coproduct is the deconcatenation map [10,17,22]. A similar structure appears here when we collect together the homology groups of orthogonal groups, with Steinberg modules coefficients, in all dimensions. This is the subject of Section 7, after a short Section 6 showing that the shuffle product already exists at the level of the Steinberg modules themselves.

Section 8 gives the connection of the previous section to scissors congruence groups. It affords a new point of view on the Hopf structure of the Sah algebra. We discover eventually two noticeable facts: the join comes from a shuffle product and the Dehn invariant reduces to a deconcatenation map.

Section 9 does not consider scissors congruence groups and can be read after Section 7, but translated into the language of scissors congruences, it is in the heart of the proof of the results of Section 10. The object of study here is a tricomplex which takes three directions into account (see (14) (18) (19)): one is given by the homology of Tits buildings, the second is related to the homology of groups, and behind the third are hidden the Dehn invariants at the level of Steinberg modules. We study the spectral sequences attached to this tricomplex. A most remarkable fact is the appearance of the cobar–bar construction [5,7,16] over the orthogonal algebra which implies the collapsing of one of these sequences, and opens the way to the results of the last section.

Throughout the paper, we argue modulo torsion, considering systematically $\mathbb{Q}$-vector spaces instead of $\mathbb{Z}$-modules.

2. Steinberg modules

2.1. Tits buildings for classical geometries

We first recall some well-known facts [3]. We consider three types of $\mathbb{R}$-vector spaces E of dimension n, corresponding respectively to spherical, hyperbolic and Euclidean geometries in dimension $n-1$. By E^s we mean a Euclidean vector space E (s will be often omitted), E^h denote an $\mathbb{R}$-vector space E with a Lorentzian metric q of type $(1, n-1)$, and E^e is an $\mathbb{R}$-vector space E equipped with a non-zero linear form λ, such that $H = \mathrm{Ker}\,\lambda$ is a Euclidean vector space. Throughout the text, $\star$ will be written for s, h or e, sometimes only for h or e.

We define a *geometric subspace* of $E^\star$ as follows. It is simply a linear subspace for E^s. For E^h it is a linear subspace containing a vector v with $q(v) = -1$. For E^e it is $\{0\}$ or a linear subspace not contained in H.

By $O(E^\star)$, $\star = s, h, e$, we mean respectively the Euclidean orthogonal group of E^s, the orthogonal group of E^h with its Lorentz form q, and the subgroup of the linear group of E^e composed of elements leaving λ^2 invariant and such that their restriction to H is orthogonal. These groups are to be thought of as the isometry groups of the sphere $S(E)$, the hyperboloid $q(x) = -1$ which is the double hyperbolic space and the double affine space $(\lambda(x))^2 = 1$. Note that the last two groups differ from the usual isometry groups of the hyperbolic space and the affine Euclidean space.

The *Tits building* $\mathscr{T}(E^\star)$ is the simplicial set given by the nerve of the category of proper non-zero geometric subspaces of $E^\star$, with morphisms given by inclusions (see [2]). In particular, a p-simplex is a flag $U_0 \subset U_1 \subset \cdots \subset U_p$, and the faces and degeneracies read

$$U_0 \subset U_1 \subset \cdots \subset U_p \mapsto U_0 \subset \cdots \subset \hat{U}_i \subset \cdots \subset U_p,$$

$$U_0 \subset U_1 \subset \cdots \subset U_p \mapsto U_0 \subset \cdots \subset U_i \subset U_i \subset \cdots \subset U_p.$$

The following result of Salomon–Tits is well-known (see Dupont [3], Theorem 3.5).

Theorem 2.1.1. *Assume that* $\dim E = n$, *then the reduced homology of the simplicial set* $\mathscr{T}(E^\star)$ *satisfies*

$$\tilde{H}_i(\mathscr{T}(E^\star), \mathbb{Q}) = 0, \qquad for\ i \neq n-2.$$

The *Steinberg module* $\mathscr{S}t(E^\star)$ is classically defined as $\tilde{H}_{n-2}(\mathscr{T}(E^\star), \mathbb{Q})$, for $n \geqslant 2$ and 0 for $\dim E = 1$. By convention, we write also $\mathscr{S}t(E^s) = \mathbb{Q}$ for $E^s = 0$. The Steinberg module $\mathscr{S}t(E^\star)$ has a natural structure of $O(E^\star)$-module.

2.2. Tits buildings and direct sum decompositions

We observe that a p-simplex of the Tits building of E^s is the same thing as an orthogonal direct sum decomposition

$$E = E_0 \oplus E_1 \oplus \cdots \oplus E_p \oplus E_{p+1},$$

where E_0 and E_{p+1} are non-zero subspaces, the relation between flags and direct sums being given by $U_i = \bigoplus_{j \leqslant i} E_j$, $i \leqslant p$. We write $(E_0, E_1, \ldots, E_p, E_{p+1})$ for the corresponding simplex. The faces and degeneracies translate to the maps

$$\varepsilon_i : (E_0, \ldots, E_{p+1}) \mapsto (E_0, \ldots, E_i \oplus E_{i+1}, \ldots, E_{p+1}), \qquad i = 0, \ldots, p,$$

$$\eta_i : (E_0, \ldots, E_{p+1}) \mapsto (E_0, \ldots, E_i, \{0\}, E_{i+1}, \ldots, E_{p+1}), \qquad i = 0, \ldots, p.$$

This description is compatible with the action of $O(E^s)$.

We point out that a *non-degenerate* p-simplex is an orthogonal decomposition $E_0 \oplus \cdots \oplus E_{p+1}$ of E by non-zero subspaces. In particular for $\dim E = n$, a non-degenerate $(n-2)$-simplex is nothing but an orthogonal decomposition of E by one-dimensional subspaces.

Considering the situation E^h of hyperbolic geometry, we note that the restriction of q to a geometric subspace of dimension l is non-degenerate of type $(1, l-1)$ and that the orthogonal of such a subspace is Euclidean. As a result, we can identify a flag $U_0 \subset U_1 \subset \cdots \subset U_p$, in the corresponding Tits building, to an orthogonal direct sum decomposition

$$E = E_0 \oplus E_1 \oplus \cdots \oplus E_p \oplus E_{p+1},$$

where E_0 is a non-zero geometric subspace, and for $i > 0$, the E_i are Euclidean with $E_{p+1} \neq 0$. We have the same description as before for faces and degeneracies. A non-degenerate $(n-2)$-simplex is an orthogonal decompositions of E by one-dimensional non-zero linear subspaces $E_0, E_1, \ldots, E_{n-1}$, with q negative (resp. positive) on E_0 (resp. $E_1, \ldots, E_{n-1}$).

The Euclidean setting E^e is a little bit different. A flag $U_0 \subset U_1 \subset \cdots \subset U_p$ can be seen as a direct sum decomposition

$$E = E_0 \oplus E_1 \oplus \cdots \oplus E_p \oplus E_{p+1},$$

where E_0 is a non-zero linear subspace not contained in H, E_{p+1} is non-zero and $(E_0 \cap H) \oplus E_1 \oplus \cdots \oplus E_p \oplus E_{p+1}$ is an orthogonal direct sum decomposition of H.

We observe that the direct sum decompositions are also compatible with the action of the group $O(E^\star)$ in the two previous cases.

2.3. Basic complexes

In what follows, we want to work in terms of non-degenerate simplices, staying in the *semi-simplicial* world, that is ignoring degeneracies. This point of view is justified by making natural later the appearance of normalized bar and cobar complexes.

Let $s\Delta$ be the category of finite sets $[n] = \{0, 1, \ldots, n\}$ with morphisms the increasing injective maps. We introduce a variant of the singular complex of the Tits building of E^s as a semi-simplicial $\mathbb{Q}$-vector space: $s\Delta^{op} \to \mathbb{Q}$-$\underline{\text{vect}}$, $[p] \to \mathscr{W}_p(E^s)$. For this, we put $\mathscr{W}_0(E^s) = \mathbb{Q}$ if $E = 0$, $\mathscr{W}_0(E^s) = 0$ otherwise and define $\mathscr{W}_p(E^s)$ to be the $\mathbb{Q}$-vector space freely generated by the symbols $(E_1, \ldots, E_p)$, where $E = E_1 \oplus \cdots \oplus E_p$ is an orthogonal direct sum decomposition by non-zero subspaces. The faces are given by the maps $\delta_0 = 0$, $\delta_p = 0$ and δ_i: $(E_1, \ldots, E_p) \mapsto (E_1, \ldots, E_i \oplus E_{i+1}, \ldots, E_p)$, for $i = 1, \ldots, p-1$.

This semi-simplicial $\mathbb{Q}$-vector space is an $O(E^s)$-module and its homology coincides with the reduced homology of the Tits building *shifted by two* (this is the double suspension), in particular $\mathscr{S}t(E^s) = H_n(\mathscr{W}_*(E^s))$, if $\dim E = n$. The associated chain complex is written $(\mathscr{W}_*(E^s), d)$.

We introduce analogous chain complexes $(\mathscr{W}_*(E^h), d)$ and $(\mathscr{W}_*(E^e), d)$ (here $\dim E > 0$), associated to the semi-simplicial $\mathbb{Q}$-vector spaces related to the Tits systems $\mathcal{T}(E^h)$ and $\mathcal{T}(E^e)$, but *shifted only by one*: the reason of this shift will appear later in Section 3.2 and is due to the presence of coefficients. Here $\mathcal{W}_p(E^\star)$, for $\star = h, e$, is the free $\mathbb{Q}$-vector space generated by the symbols $(E_0, \dots, E_p)$, where $E = E_0 \oplus \cdots \oplus E_p$ is the direct sum decomposition associated to a non-degenerate flag $U_0 \subset U_1 \subset \cdots \subset U_{p-1}$ of $E^\star$. It is important to write the first subspace E_0 and not E_1: see the spectral sequences in Section 4.2. With this description, $\mathcal{S}t(E^\star) = H_{n-1}(\mathcal{W}_*(E^\star))$, for $\dim E = n > 0$.

2.4. Weights for Steinberg modules in spherical geometry

We prove in this section that Adams operations as considered in [9] induce a splitting of the spherical Steinberg modules: actually semi-simplicial structures are sufficient to get Adams operations.

For this purpose, paraphrasing *loc.cit.*, we define $s\mathrm{Fin}'$ as the category of finite pointed sets, with *surjective* pointed maps as morphisms. A $s\mathrm{Fin}'$-module is a semi-simplicial $\mathbb{Q}$-vector space: $s\Delta^{op} \to \mathbb{Q}\text{-}\underline{\mathrm{vect}}$, which factorizes through the functor $s\Delta^{op} \to s\mathrm{Fin}'$, restriction of the functor $\Delta^{op} \to \mathrm{Fin}'$, defined in *loc.cit.* Section 2.1.

Proposition 2.4.1. *The semi-simplicial space $\mathcal{W}_*(E^s)$ has a natural structure of $s\mathrm{Fin}'$-space.*

In fact staying with the Tits building, we would have a Fin'-structure, but the consideration of degeneracies is artificial for decompositions. For any pointed surjective map $f:[p] \to [q]$, one needs to define a good map $\bar{f}: \mathcal{W}_p(E^s) \to \mathcal{W}_q(E^s)$. If $f^{-1}(0) = \{0\}$ we take $\bar{f}(E_1, \dots, E_p) = (F_1, \dots, F_q)$, where $F_j = \bigoplus_{i \in f^{-1}(j)} E_i$, otherwise we put $\bar{f} = 0$. One checks that this is compatible with the functor $s\Delta^{op} \to s\mathrm{Fin}'$.

As a result, we get (see the end of [9] Section 3.4, for more details)

Corollary 2.4.2. *The Steinberg module $\mathcal{S}t(E^s)$ splits, as a direct sum of $O(E^s)$-submodules*

$$\bigoplus_{r=1}^{n} \mathcal{S}t^{(r)}(E^s),$$

given by the weight decomposition, under the Adams operations coming from the $s\mathrm{Fin}'$-structure.

In fact since the singular complex $(\mathcal{W}_*(E^s), d)$ is a complex of $\mathbb{Q}$-module, it splits as a direct sum of subcomplexes of $O(E^s)$-modules under projections induced by the action of the Eulerian idempotents: the point is that the $s\mathrm{Fin}'$-structure is enough to assure the compatibility of these projections with the differentials. The Eulerian idempotents $e_n^{(r)}$ are elements of the group algebras over $\mathbb{Q}$ of the symmetric groups which act here on orthogonal decompositions of E, by permutation of the factors. The image of the projection given by $e_n^{(r)}$ is precisely $\mathcal{S}t^{(r)}(E^s)$.

2.5. Notations

We fix notations in canonical situations.

If E^s is $\mathbb{R}^n$ with the canonical inner product, we put $\mathscr{T}(n)$ for $\mathscr{T}(E^s)$, $\mathscr{S}t(n)$ for $\mathscr{S}t(E^s)$, $O(n)$ for the orthogonal group $O(E^s)$ and $\mathscr{W}_*(n)$ for $\mathscr{W}_*(E^s)$.

If E^h is $\mathbb{R}^n$ with the quadratic form $-x_0^2+x_1^2+\cdots+x_{n-1}^2$, we write $\mathscr{T}^h(n)$ for $\mathscr{T}(E^h)$, $\mathscr{S}t^h(n)$ for $\mathscr{S}t(E^h)$, $O(1,n-1)$ for the group $O(E^h)$ and $\mathscr{W}_*^h(n)$ for $\mathscr{W}_*(E^h)$. We use also the usual notation $O^+(1,n)$ for the *isometry group* of the hyperbolic $(n-1)$-space $\{x\colon -x_0^2+x_1^2+\cdots+x_{n-1}^2=-1, x_0>0\}$, which is a subgroup of index 2 of $O(1,n-1)$.

If E^e is $\mathbb{R}^n$ with $\{x_0=0\}$ equipped with the canonical inner product, we put $\mathscr{T}^e(n)$ for $\mathscr{T}(E^e)$, $\mathscr{S}t^e(n)$ for $\mathscr{S}t(E^e)$, $E(1,n-1)$ for $O(E^e)$ and $\mathscr{W}_*^e(n)$ for $\mathscr{W}_*(E^e)$. We use the notation $E(n-1)$ for the *isometry group* of the affine $(n-1)$-Euclidean space, which appears as a subgroup of index 2 of $E(1,n-1)$.

3. The orthogonal algebra

3.1. The orthogonal algebra and related differential graded algebras

For a group of isometries G and a G-module M, we denote by M^t the module $M\otimes\mathbb{Q}^t$, where $\mathbb{Q}^t$ is the "orientation" module. By convention, $O(0)$ is the trivial group. We put

$$\mathbf{A}=\bigoplus_{n\geqslant 0, i\geqslant 0}\mathbf{A}_{n,i},$$

with

$$\mathbf{A}_{n,i}:=H_i(O(n),\mathbb{Q}^t),\qquad n\geqslant 0,\ i\geqslant 0.$$

Clearly

$$\mathbf{A}_{0,0}:=\mathbb{Q},\ \mathbf{A}_{0,i}:=0,\quad i\geqslant 1,\qquad \mathbf{A}_{n,0}:=0,\quad n\geqslant 1.$$

We have also $\mathbf{A}_{n,i}=0$ for n odd. In fact the homology groups $H_*(O(n),\mathbb{Q}^t)$ are 0 for n odd, by the "center kills" lemma (see for example [3], Lemma 5.4).

The bigraded $\mathbb{Q}$-vector space $\mathbf{A}$ is provided with the structure of a bigraded algebra whose product is given by the composition of the following maps

$$H_i(O(m),\mathbb{Q}^t)\otimes H_j(O(n),\mathbb{Q}^t)\xrightarrow{\phi}H_{i+j}(O(m)\times O(n),\mathbb{Q}^t\otimes\mathbb{Q}^t)$$

$$\xrightarrow{\psi}H_{i+j}(O(m+n),\mathbb{Q}^t).$$

Here ϕ comes from Eilenberg–Zilber–Künneth, and ψ from the diagonal embedding $O(m)\times O(n)\subset O(m+n)$. We call this algebra $\mathbf{A}$ the *orthogonal algebra*.

Lemma 3.1.1. *The orthogonal algebra* $\mathbf{A}$ *is graded-commutative.*

In other word $a.b=(-1)^{(i+m)(j+n)}b.a$, if $a\in\mathbf{A}_{m,i}$ and $b\in\mathbf{A}_{n,j}$. This comes from two remarks.

The natural diagram

$$\begin{array}{ccc} H_i(O(m),\mathbb{Q}^t)\otimes H_j(O(n),\mathbb{Q}^t) & \xrightarrow{\phi} & H_{i+j}(O(m)\times O(n),\mathbb{Q}^t\otimes\mathbb{Q}^t) \\ \downarrow & & \downarrow \\ H_j(O(n),\mathbb{Q}^t)\otimes H_i(O(m),\mathbb{Q}^t) & \xrightarrow{\phi} & H_{i+j}(O(n)\times O(m),\mathbb{Q}^t\otimes\mathbb{Q}^t) \end{array}$$

commutes up to the sign $(-1)^{ij}$.

Moreover the natural diagram

$$\begin{tikzcd} H_{i+j}(O(m)\times O(n),\mathbb{Q}^t\otimes\mathbb{Q}^t) \arrow[dd, "\tau"'] \arrow[dr, "\psi"] & \\ & H_{i+j}(O(m+n),\mathbb{Q}^t) \\ H_{i+j}(O(n)\times O(m),\mathbb{Q}^t\otimes\mathbb{Q}^t) \arrow[ur, "\psi'"'] & \end{tikzcd}$$

is commutative. In fact, let g_σ be the permutation matrix related to

$$\sigma=(m+1,m+2,\ldots,m+n,1,2,\ldots,m),$$

the maps $\psi'\circ\tau$ and ψ are related by the conjugation by g_σ. Since $\mathbf{A}$ is concentrated in even dimensional degrees, $\mathrm{sign}(\sigma)=(-1)^{mn}=1$ which implies $\psi'\circ\tau=\psi$ (see [1] Chapter III, Section 8).

The orthogonal algebra $\mathbf{A}$ is connected and has a split augmentation $\varepsilon:\mathbf{A}\to\mathbb{Q}$: we write $\mathbf{A}=\bar{\mathbf{A}}\oplus\mathbb{Q}$, with $\bar{\mathbf{A}}=\bigoplus_{(n,i)\neq(0,0)}\mathbf{A}_{n,i}$ the augmentation ideal.

We will work in the setting of *Differential Homological Algebra* [15], to keep track of the product structures in some spectral sequences. In this respect we consider the orthogonal algebra as a graded commutative differential algebra, with zero differential. The dimension n will appear as a weight.

Here are some related algebras. Let $(C_*(O(n),\mathbb{Q}^t),\delta)$ be the standard non-homogeneous complex of the homology of groups. The bigraded differential module

$$(\mathbf{C},\delta)=\left(\bigoplus_{n,i}C_i(O(n),\mathbb{Q}^t),\delta\right),$$

is an algebra for the product given by the composition

$$C_i(O(m,\mathbb{Q}^t)\otimes C_j(O(n),\mathbb{Q}^t)\to C_{i+j}(O(m)\times O(n,),\mathbb{Q}^t\otimes\mathbb{Q}^t)$$
$$\to C_{i+j}(O(m+n),\mathbb{Q}^t),$$

the first map being the Eilenberg–Zilber shuffle map. As a differential graded algebra, $\mathbf{A}$ coincides with the homology of $\mathbf{C}$.

The algebra $\mathbf{C}$ is not graded commutative, so we introduce another algebra which is graded commutative and quasi-isomorphic to $\mathbf{C}$. For this purpose, let Σ_n be the symmetric group, identified to the subgroup of permutation matrices in $O(n)$, and let

$C_i(O(n),\mathbb{Q}^t)_{\Sigma_n}$ be the coinvariants under the action stemming from conjugation ([1] Chapter III, Section 8), we define

$$\mathbf{C}_\Sigma = \bigoplus_{n,i} C_i(O(n),\mathbb{Q}^t)_{\Sigma_n},$$

Proposition 3.1.2. *With the product induced by the shuffle product, the differential algebra $\mathbf{C}_\Sigma$ is graded commutative and the natural projection $\mathbf{C} \to \mathbf{C}_\Sigma$ is a quasi-isomorphism of differential algebras.*

The induced action of Σ_n in $H_*(O(n),\mathbb{Q}^t)$ is trivial, so the proof is a consequence of the following fact: let G be a finite group acting on a complex (C_*,d) of $\mathbb{Q}$-vector spaces, if the induced action of G on $H_*(C)$ is trivial, the morphism $C \to C_G$ is a quasi-isomorphism.

3.2. Bar constructions

In this paper, we will encounter the normalized bar (resp. cobar) construction on different differential algebras (resp. coalgebras): see the Ref. [5], Section 2, or [16,15,14], for these constructions.

In this subsection, we address the bar construction: it goes from the category of differential graded augmented algebras to the category of differential graded supplemented coalgebras, and associates to a differential algebra A a differential coalgebra noted $\mathscr{B}A$. In particular in what follows, we will consider the bar constructions over the algebras $\mathbf{A}$, $\mathbf{C}$ and $\mathbf{C}_\Sigma$.

Let $(A = \bar{A} \oplus \mathbb{Q}, \delta)$ a differential graded augmented $\mathbb{Q}$-algebra, as a graded module, the reduced bar construction $\mathscr{B}A$ is the tensor algebra $\mathbf{T}(\mathrm{s}\bar{A})$ over the suspension of $\bar{A}$. The reduced coproduct reads

$$\bar{\Delta}([\mathrm{s}a_1|\dots|\mathrm{s}a_p]) = \sum_{i=1}^{p-1} [\mathrm{s}a_1|\dots|\mathrm{s}a_i] \otimes [\mathrm{s}a_{i+1}|\dots|\mathrm{s}a_p].$$

As usual, $[a_1|\dots|a_p]$ denotes the element $a_1 \otimes \cdots \otimes a_p$ in the tensor algebra. The differential is given by $D = d + \delta$, where

$$d([\mathrm{s}a_1|\dots|\mathrm{s}a_p]) = \sum_{i=1}^{p-1} \varepsilon_i [\mathrm{s}a_1|\dots|\mathrm{s}a_i a_{i+1}|..|\mathrm{s}a_p],$$

$$\delta([a_1|\dots|a_p]) = -\sum_{i=1}^{p} \varepsilon_i [\mathrm{s}a_1|\dots|\mathrm{s}\delta a_i|..|\mathrm{s}a_p],$$

with the sign $\varepsilon_i = (-1)^{\sum_{j\leqslant i} \deg \mathrm{s}a_j}$.

The homology of the bar complex $(\mathscr{B}A, D)$ is $Tor^A(\mathbb{Q},\mathbb{Q})$, where $\mathbb{Q}$ is the trivial A-bimodule, that is $\mathbb{Q}$ is placed in degree 0, and A acts on both side by the scalar multiplication through the quotient $A/\bar{A} = \mathbb{Q}$. The bar complex appears as the total complex of a bicomplex, where the element $[\mathrm{s}a_1|\dots|\mathrm{s}a_p]$ of total degree $p + \sum \deg a_i$, is of bidegree (p,q) with $q=\sum \deg a_i$. We get from this bicomplex the *canonical* spectral sequence [4,12]

$$E^2_{p,q}=Tor^{H_*(A)}_{p,q}(\mathbb{Q},\mathbb{Q})\Rightarrow Tor^A_{p+q}(\mathbb{Q},\mathbb{Q}).$$

When A is graded commutative, $\mathscr{B}A$ has richer structure. It is a differential Hopf algebra: the product is given by the shuffle product in the tensor algebra. In this case the previous spectral sequence is a spectral sequence of Hopf algebras. In particular, for $A=\mathbf{C}_\Sigma$, we get the spectral sequence of Hopf algebras (see for example the chapter on Eilenberg–Moore spectral sequence in [12])

$$E^2_{p,q}=Tor^{\mathbf{A}}_{p,q}(\mathbb{Q},\mathbb{Q})\Rightarrow Tor^{\mathbf{C}_\Sigma}_{p+q}(\mathbb{Q},\mathbb{Q}).$$

Since we want to use Hodge decompositions for Hochschild homology, we remark that $Tor^A(\mathbb{Q},\mathbb{Q})$ is nothing else than the *Hochschild homology* $H_*(A,\mathbb{Q})$ of A with coefficients in the trivial bimodule $\mathbb{Q}$ [11,24]. In the case of the algebra with zero differential $\mathbf{A}$, these groups are naturally bigraded and we will denote the pieces in dimension n of the bigrading by $H_{p,q}(\mathbf{A},\mathbb{Q})_n$. This corresponds to the following pieces of the reduced bar complex of $\mathbf{A}$

$$\mathscr{B}_{p,q}(\mathbf{A})_n=\bigoplus_{q_1+\cdots+q_p=q}^{n_1+\cdots+n_p=n} H_{q_1}(O(n_1),\mathbb{Q}^t)\otimes\cdots\otimes H_{q_p}(O(n_p),\mathbb{Q}^t),$$

where the n_j are non-zero, in fact even integers. We note that if we forget the suspensions, we can take as differential

$$d(a_1\otimes\cdots\otimes a_p)=\sum_{i=1}^{p-1}(-1)^{i+1}a_1\otimes\cdots\otimes a_ia_{i+1}\otimes\cdots\otimes a_p.$$

Notice also that $H_{1,*}(\mathbf{A},\mathbb{Q})=\bar{\mathbf{A}}/\bar{\mathbf{A}}^2$ is the indecomposable part of $\mathbf{A}$, and $H_{1,q}(\mathbf{A},\mathbb{Q})_n$ is a quotient of the homology group $H_q(O(n),\mathbb{Q}^t)$, which we call the *indecomposables* of $H_q(O(n),\mathbb{Q}^t)$.

From Vigué-Poirrier [23] Section 1 (in fact one has to extend the results in here to the case of symmetric bimodules, which is easy; see also [9,18] for the non-graded case) we know that the Hochschild homology of a differential graded-*commutative* $\mathbb{Q}$-algebra, with coefficients in a *symmetric* bimodule, admits Hodge decompositions, induced by the action of Euler idempotents: these decompositions are of the type

$$H_l(A,\mathbb{Q})=\bigoplus_{r=1}^{l}H_l^{(r)}(A,\mathbb{Q}).$$

For the algebra $\mathbf{A}$, with zero differential, these decompositions are compatible with the bigrading of $H_l(\mathbf{A},\mathbb{Q})$, and we have in fact

$$H_{p,q}(\mathbf{A},\mathbb{Q})=\bigoplus_{r=1}^{p}H^{(r)}_{p,q}(A,\mathbb{Q}).$$

If we look at the differential graded commutative algebra $\mathbf{C}_\Sigma$, the bar complex splits under the action of Euler idempotents, into a direct sum of sub-complexes as in [23,9]. This gives spectral sequences, one for each (n,r) (see also Theorem 4.1.2)

$$E^2_{p,q}=H^{(r)}_{p,q}(\mathbf{A},\mathbb{Q})_n\Rightarrow H^{(r)}_{p+q}(\mathbf{C}_\Sigma,\mathbb{Q})_n.$$

We end this section by a useful result on the bar complex of $\mathbf{A}$. First the following is implicit in [2]:

Proposition 3.2.1. *For i even, $H_i(O(2),\mathbb{Q}^t)=0$.*
For i odd,

$$H_i(O(2),\mathbb{Q}^t)\cong H_i(SO(2),\mathbb{Q})\cong \bigwedge_{\mathbb{Q}}^{i}(\mathbb{R}/\mathbb{Q}).$$

Proof. Since $SO(2)$ is commutative, it is known that

$$H_i(SO(2),\mathbb{Q})\cong \bigwedge_{\mathbb{Q}}^{i}(SO(2)\otimes\mathbb{Q}).$$

Moreover $SO(2)\otimes\mathbb{Q}\cong\mathbb{R}/\mathbb{Z}\otimes\mathbb{Q}\cong\mathbb{R}/\mathbb{Q}$. To complete the proof, one considers the spectral sequence of the extension

$$0\to SO(2)\to O(2)\to\{\pm 1\}\to 0,$$

and use the fact that the action of $\{\pm 1\}$ on $SO(2)$ arising from this extension is given by $g\mapsto g^{-1}$. $\square$

We deduce the corollary:

Corollary 3.2.2. *Let n be even,*

(i) *for $p>n/2$, $\mathscr{B}_{p,q}(\mathbf{A})_n=0$,*
(ii) *if $n/2$ is even and q is odd, $\mathscr{B}_{n/2,q}(\mathbf{A})_n=0$,*
(iii) *if $n/2$ is odd and q is even, $\mathscr{B}_{n/2,q}(\mathbf{A})_n=0$.*

3.3. Modules over the orthogonal algebra

We introduce two right-modules over the orthogonal algebra $\mathbf{A}$, corresponding respectively to hyperbolic and Euclidean geometry.

We consider the bigraded $\mathbb{Q}$-vector space

$$\mathbf{M}^h=\bigoplus_{n\geqslant 0,i\geqslant 0}\mathbf{M}^h_{n,i},$$

where $\mathbf{M}^h_{n,i}:=H_i(O(1,n-1),\mathbb{Q}^t)$, $n\geqslant 1$, $i\geqslant 0$ and $\mathbf{M}^h_{0,i}=0$. This vector space acquires a natural structure of bigraded right-$\mathbf{A}$-module by the maps

$$H_i(O(1,m-1),\mathbb{Q}^t)\otimes H_j(O(n),\mathbb{Q}^t)\to H_{i+j}(O(1,m+n-1),\mathbb{Q}^t),$$

arising from cross products and diagonal inclusions

$$O(1,m-1)\times O(n)\subset O(1,m+n-1).$$

Similarly we have an $\mathbf{A}$-module

$$\mathbf{M}^e=\bigoplus_{n\geqslant 0,i\geqslant 0}\mathbf{M}^e_{n,i},$$

where $\mathbf{M}^e_{n,i} := H_i(E(1,n-1),\mathbb{Q}^t)$, $n \geqslant 1$, $i \geqslant 0$ and $\mathbf{M}^e_{0,i} = 0$. The bigraded $\mathbf{A}$-module structure stems from the maps

$$H_i(E(1,m-1),\mathbb{Q}^t)\otimes H_j(O(n),\mathbb{Q}^t) \to H_{i+j}(E(1,m+n-1),\mathbb{Q}^t),$$

arising from cross products and diagonal inclusions

$$E(1,m-1)\times O(n) \subset E(1,m+n-1).$$

By the "center kills" lemma, the homology groups $H_*(O(1,n-1),\mathbb{Q}^t)$ and $H_*(E(1,n-1),\mathbb{Q}^t)$ are 0, if n is odd.

We have the bar construction for modules: it associates to a differential A-module M a differential right $\mathscr{B}A$-comodule $\mathscr{B}(M,A)$. See [5] for the precise formulations. In the setting of differential algebra, the homology of $\mathscr{B}(M,A)$ is $Tor^A(M,\mathbb{Q})$ which is actually the Hochschild homology $H_*(A,M)$ of A with coefficients in the A-module M, considered as a bimodule by the trivial action on the left.

We consider the modules $\mathbf{M}^\star$ as graded with zero differential. As in the previous section, the Hochschild homology $H_p(\mathbf{A},\mathbf{M}^\star)$ is naturally bigraded and the piece $H_{p,q}(\mathbf{A},\mathbf{M}^\star)_n$ corresponds to the following piece of the reduced bar complex $\mathscr{B}(\mathbf{M}^\star,\mathbf{A})$

$$\mathscr{B}_{p,q}(\mathbf{M}^h,\mathbf{A})_n = \bigoplus_{\substack{n_0+n_1+\cdots+n_p=n \\ q_0+q_1+\cdots+q_p=q}} H_{q_0}(O(1,n_0-1),\mathbb{Q}^t)\otimes H_{q_1}(O(n_1),\mathbb{Q}^t)\otimes\cdots\otimes H_{q_p}(O(n_p),\mathbb{Q}^t),$$

and

$$\mathscr{B}_{p,q}(\mathbf{M}^e,\mathbf{A})_n = \bigoplus_{\substack{n_0+n_1+\cdots+n_p=n \\ q_0+q_1+\cdots+q_p=q}} H_{q_0}(E(1,n_0-1),\mathbb{Q}^t)\otimes H_{q_1}(O(n_1),\mathbb{Q}^t)\otimes\cdots\otimes H_{q_p}(O(n_p),\mathbb{Q}^t).$$

Here the integers n_j are assumed to be non-zero: only even n_j are relevant.

The following results are similar to those of Proposition 3.2.1 and Corollary 3.2.2.

Proposition 3.3.1. *Let $SO(1,1)$ and $SE(1,1)$ be the kernels of the determinant maps $O(1,1)\to\{\pm 1\}$ and $E(1,1)\to\{\pm 1\}$*

(i) *For i even, $H_i(O(1,1),\mathbb{Q}^t)=0$, and for i odd*

$$H_i(O(1,1),\mathbb{Q}^t)\cong H_i(SO(1,1),\mathbb{Q})\cong \bigwedge\nolimits_{\mathbb{Q}}^{i}\mathbb{R}.$$

(ii) *For i even, $H_i(E(1,1),\mathbb{Q}^t)=0$, and for i odd*

$$H_i(E(1,1),\mathbb{Q}^t)\cong H_i(SE(1,1),\mathbb{Q})\cong \bigwedge\nolimits_{\mathbb{Q}}^{i}\mathbb{R}.$$

The proof is similar to the one of Proposition 3.2.1, with a minor change: using the notation of Section 2.5, we observe that $SO(1,1)=SO^+(1,1)\times\{\pm Id\}$ and $SE(1,1)=SE(1)\times\{\pm Id\}$. On the other hand it is well known that $SO^+(1,1)$ is isomorphic to $\mathbb{R}$ and $SE(1)$ is the group of translation of the affine line.

Corollary 3.3.2. *Let n be even*,

(i) *for* $p\geqslant n/2$, $\mathscr{B}_{p,q}(\mathbf{M}^\star,\mathbf{A})_n=0$,
(ii) *if $n/2$ is even and q is odd*, $\mathscr{B}_{(n/2-1),q}(\mathbf{M}^\star,\mathbf{A})_n=0$,
(iii) *if $n/2$ is odd and q is even*, $\mathscr{B}_{(n/2-1),q}(\mathbf{M}^\star,\mathbf{A})_n=0$.

4. Homology with Steinberg modules coefficients and the bar construction

4.1. The spherical case

As before, the homology groups

$$H_i(O(n),\mathscr{S}t(n)^t)$$

are 0 for n odd, by “center kills”. We want to relate these groups, for n even, to the bar construction over the algebra $\mathbf{A}$. The connection is given by a spectral sequence:

Theorem 4.1.1. *For n even, there is a spectral sequence*

$$E^2_{p,q}=H_{p,q}(\mathbf{A},\mathbb{Q})_n\Rightarrow H_{p+q-n}(O(n),\mathscr{S}t(n)^t),$$

with $E^2_{p,q}=0$ for $p>n/2$, $E^2_{n/2,q}=0$ for $n/2$ even and q odd, or $n/2$ odd and q even.

We start from the hyperhomology groups

$$\mathbb{H}_*(O(n),\mathscr{W}_*(n)^t).$$

These are the homology groups of the double complex

$$\Gamma_n=(C_s(O(n),\mathscr{W}_r(n)^t),d,\delta), \tag{1}$$

where C_* means the non-homogeneous chain complex of the homology of groups with differential δ, and d is the differential of $\mathscr{W}_*(n)$.

The spectral sequence of this bicomplex, associated to the filtration by degree s, collapses from E^1, since $E^1_{p,q}=C_p(O(n),H_q(\mathscr{W}_*(n))^t)$, that is $E^1_{p,q}=0$ for $q\neq n$, and $E^1_{p,n}=C_p(O(n),\mathscr{S}t(n)^t)$. We get $E^2_{p,n}=E^\infty_{p,n}=H_p(O(n),\mathscr{S}t(n)^t)$, and then

$$\mathbb{H}_i(O(n),\mathscr{W}_*(n)^t)\cong H_{i-n}(O(n),\mathscr{S}t(n)^t).$$

We now address the second spectral sequence of (1). For this sequence, $E^1_{p,q}=H_q(O(n),\mathscr{W}_p(n)^t)$. One key fact is: the $O(n)$-module $\mathscr{W}_p(n)^t$ splits as a direct sum of induced modules, along the orbits for the action of $O(n)$ into the set of direct orthogonal decompositions of $\mathbb{R}^n$ by p non-zero subspaces. Since these orbits are parametrized by the p-tuples $(n_1,\dots,n_p)$ of non-zero integers with $n_1+\cdots+n_p=n$, and the stabilizer of the associated canonical decomposition is $O(n_1)\times\cdots\times O(n_p)$, an application of Shapiro's lemma, [1, Chapter III] gives an isomorphism

$$H_q(O(n),\mathscr{W}_p(n)^t)\cong\bigoplus_{n_1+\cdots+n_p=n}H_q(O(n_1)\times\cdots\times O(n_p),\mathbb{Q}^t).$$

By the Künneth formula, we finally obtain

$$E^1_{p,q}\cong\bigoplus_{\substack{q_1+\cdots+q_p=q\\ n_1+\cdots+n_p=n}}H_{q_1}(O(n_1),\mathbb{Q}^t)\otimes\cdots\otimes H_{q_p}(O(n_p),\mathbb{Q}^t).$$

We check that, through the isomorphisms coming from Shapiro's lemma, the differential d^1 is the differential of the bar complex $\mathscr{B}_*(\mathbf{A})$ of Section 3.2. As a direct sum of induced modules, $\mathscr{W}_p(n)^t$ is canonically isomorphic to

$$\bigoplus_{n_1+\cdots+n_p=n}\mathbb{Z}O(n)\otimes_{\mathbb{Z}[O(n_1)\times\cdots\times O(n_p)]}\mathbb{Q}^t,$$

if we take as base points of the orbits the canonical orthogonal decompositions of $\mathbb{R}^n$ associated to the partitions of the canonical basis of $\mathbb{R}^n$. The Schapiro isomorphism (see for example [1, Chapter III, Section 8, Exercise 2] arises from the morphism of pairs

$$\begin{aligned}&(O(n_1)\times\cdots\times O(n_p),\mathbb{Z}[O(n_1)\times\cdots\times O(n_p)]\otimes_{\mathbb{Z}[O(n_1)\times\cdots\times O(n_p)]}\mathbb{Q}^t)\\ &\quad\to(O(n),\mathbb{Z}O(n)\otimes_{\mathbb{Z}[O(n_1)\times\cdots\times O(n_p)]}\mathbb{Q}^t).\end{aligned}$$

So it suffices to remark that there are commutative diagrams of modules

$$\begin{tikzcd}
\mathbb{Z}O(n)\otimes_{\mathbb{Z}[O(n_1)\times\cdots\times O(n_i)\times O(n_{i+1})\times\cdots\times O(n_p)]}\mathbb{Q}^t \arrow[r] \arrow[d] & \mathscr{W}_p(n)^t \arrow[d] \\
\mathbb{Z}O(n)\otimes_{\mathbb{Z}[O(n_1)\times\cdots\times O(n_i+n_{i+1})\times\cdots\times O(n_p)]}\mathbb{Q}^t \arrow[r] & \mathscr{W}_{p-1}(n)^t,
\end{tikzcd}$$

where the right vertical map is the ith face of $\mathscr{W}_*$. We conclude that

$$E^2_{p,q}\cong H_{p,q}(\mathbf{A},\mathbb{Q})_n.$$

We remark that the higher differentials in the last spectral sequence are kind of generalized Massey products [8].

The last assertion of the theorem is a rewriting of Corollary 3.2.2.

Remark. The direct sum of the previous spectral sequences for every n can be identified with the spectral sequence of Section 3.2

$$E^2_{p,q}=Tor^{\mathbf{A}}_{p,q}(\mathbb{Q},\mathbb{Q})\Rightarrow Tor^{\mathbf{C}}_{p+q}(\mathbb{Q},\mathbb{Q}).$$

In fact the previous proof shows that there is a quasi-isomorphism compatible with filtrations between the complexes $Tot\oplus_n\Gamma_n$ and $(\mathscr{B}\mathbf{C},D)$. We will come back to this point of view in Proposition 7.1.2.

Using the weights of Section 2.4, for the Steinberg modules, and the Hodge decompositions of Hochschild homology, the spectral sequence of the previous theorem splits as a direct sum of n sub-spectral sequences, and we obtain the more precise result:

Theorem 4.1.2. *For $r = 1,\dots,n$, there is a spectral sequence*

$$^{(r)}E^2_{p,q} = H^{(r)}_{p,q}(\mathbf{A},\mathbb{Q})_n \Rightarrow H_{p+q-n}(O(n),\mathscr{S}t^{(r)}(n)^t).$$

The complex $\mathscr{W}_*(n)$ splits under the action of Euler idempotents as a direct sum of subcomplexes $\mathscr{W}^{(r)}_*(n)$. This induces a splitting of the bicomplex (1) into bicomplexes $\Gamma^{(r)}_n = (C_s(O(n),\mathscr{W}^{(r)}_l(n)^t),d,\delta)$. The homology of $Tot\Gamma^{(r)}_n$ is $H_{*-n}(O(n),\mathscr{S}t^{(r)}(n)^t)$. One needs to show that the spectral sequence for the filtration by l of $\Gamma^{(r)}_n$ has $E^2_{p,q} = H^{(r)}_{p,q}(\mathbf{A},\mathbb{Q})_n$.

As a first step we replace Γ_n by the bicomplex of coinvariants

$$\Gamma_\Sigma = (C_s(O(n),\mathscr{W}_l(n)^t)_{\Sigma_n},\delta,d),$$

where the action of $\sigma\in\Sigma_n$ comes from the usual automorphism of pair [1, Chapter III Section 8]

$$(O(n),\mathscr{W}_l(n)^t) \to (O(n),\mathscr{W}_l(n)^t)$$

$$(A,W) \mapsto (M_\sigma A M_\sigma^{-1}, M_\sigma W),$$

with $A\in O(n)$, $W\in\mathscr{W}_l(n)$ and $M_\sigma\in O(n)$ the permutation matrix associated to σ. The natural map $\Gamma\to\Gamma_\Sigma$ induces an isomorphism from $E^2_{*,*}$ of the corresponding spectral sequences.

We look at the composition

$$\bigoplus_{i_1+\cdots+i_p=i}^{n_1+\cdots+n_p=n} C_{i_1}(O(n_1),\mathbb{Q}^t)_{\Sigma_{n_1}}\otimes\cdots\otimes C_{i_p}(O(n_p),\mathbb{Q}^t)_{\Sigma_{n_p}}$$

$$\xrightarrow{f} \bigoplus_{n_1+\cdots+n_p=n} C_i(O(n_1)\times\cdots\times O(n_p),\mathbb{Q}^t)_{\Sigma_{n_1}\times\cdots\times\Sigma_{n_p}}$$

$$\xrightarrow{g} C_i(O(n),\mathscr{W}_p(n)^t)_{\Sigma_n},$$

where the first map is given by the Eilenberg–Zilber shuffle maps and the second one comes from the canonical morphisms of pairs, as in the proof of the previous theorem.

We consider now two actions of the Euler idempotents in the source and the target of $g\circ f$. The first action will correspond to the decomposition of Hochschild homology of $\mathbf{A}$ and the second one to the s-Fin'-structure of $\mathscr{W}_*(n)$. These actions are in fact determined by the following actions of Σ_p: $\tau\in\Sigma_p$ acts on $\oplus C_{i_1}(O(n_1),\mathbb{Q}^t)_{\Sigma_{i_1}}\otimes\cdots\otimes C_{i_p}(O(n_p),\mathbb{Q}^t)_{\Sigma_{i_p}}$ (a piece of the bar complex of C_Σ) by permutation of the factors, with a sign such that the map f is equivariant for the action of τ on $\oplus C_i(O(n_1)\times\cdots\times O(n_p),\mathbb{Q}^t)_{\Sigma_{i_1}\times\cdots\times\Sigma_{i_p}}$ given by permutation of the factors. Actually the sign is the one introduced in [23, Section 1], since we are in the graded commutative case. On the other hand, τ acts on $C_i(O(n), \mathscr{W}_p(n)^t)_{\Sigma_n}$ by permutation of the components of the orthogonal direct sum decompositions of $\mathbb{R}^n$ into p factors.

To complete the proof, we have to show that the map g commutes with the actions of τ: here is the argument. Let $A_i \in O(n_i)$, $i = 1, \ldots, p$, and denote by $[A_1, \ldots, A_p]$ the corresponding "diagonal" matrix in $O(n)$. Let $\sigma \in \Sigma_n$, such that the permutation matrix M_σ satisfies

$$M_\sigma[A_1, \ldots, A_p]M_\sigma^{-1} = [A_{\tau^{-1}(1)}, \ldots, A_{\tau^{-1}(p)}],$$

and denote by $(E_1, \ldots, E_p)$ the element of $\mathscr{W}_p(n)$ associated to the partition of the canonical basis e_i, $i = 1, \ldots, n$ of $\mathbb{R}^n$

$$(e_1, \ldots, e_{n_1})(e_{n_1+1}, \ldots, e_{n_1+n_2}) \ldots (e_{n_1+\cdots+n_{p-1}}, \ldots, e_n),$$

and by $(F_1, \ldots, F_p)$, the one associated to the partition

$$(e_1, \ldots, e_{n_{\tau^{-1}(1)}})(e_{n_{\tau^{-1}(1)}+1}, \ldots, e_{n_{\tau^{-1}(1)}+n_{\tau^{-1}(2)}}) \ldots (e_{n_{\tau^{-1}(1)}+\cdots+n_{\tau^{-1}(p-1)}}, \ldots, e_n).$$

Taking into account the fact that we have taken the coinvariants under the action of Σ_n in the target, the equivariance of g follows from the relations

$$\begin{aligned} M_\sigma^{-1}(F_1, \ldots, F_p) &= (M_\sigma^{-1}(F_1), \ldots, M_\sigma^{-1}(F_p)) = \tau(E_1, \ldots, E_p) \\ &= (E_{n_{\tau^{-1}(1)}}, \ldots, E_{n_{\tau^{-1}(p)}}). \end{aligned}$$

Corollary 4.1.3. *For n even and $r > n/2$, the homology groups*

$$H_*(O(n), \mathscr{S}t^{(r)}(n)^t),$$

are trivial.

The corollary is a consequence of two remarks. By Corollary 3.2.2, ${}^{(r)}E^2_{p,q} = 0$ for $p > n/2$. On the other hand, the pieces of the Hodge decomposition $H^{(r)}_{p,q}(\mathbf{A}, \mathbb{Q})$ are 0, for $r > p$. This implies that ${}^{(r)}E^2_{p,q} = 0$, for $r > n/2$ and $p \leqslant n/2$, so that eventually ${}^{(r)}E^2 = 0$, for $r > n/2$, and the spectral sequence is trivial.

4.2. The hyperbolic and Euclidean cases

We are interested in the homology groups

$$H_i(O(1, n-1), \mathscr{S}t^h(n)^t) \quad \text{and} \quad H_i(E(1, n-1), \mathscr{S}t^e(n)^t).$$

Since by "center kills" these groups are 0 for n odd, n is always even in what follows.

Theorem 4.2.1. *There is a spectral sequence*

$$E^2_{p,q} = H_{p,q}(\mathbf{A}, \mathbf{M}^h)_n \Rightarrow H_{p+q-n+1}(O(1, n-1), \mathscr{S}t^h(n)^t),$$

with $E^2_{p,q} = 0$ for $p \geqslant n/2$ and $E^2_{n/2-1,q} = 0$ for $n/2$ even and q odd, or $n/2$ odd and q even.

We follow the procedure of the proof of Theorem 4.1.1. The hyperhomology groups

$$\mathbb{H}_*(O(1,n-1),\mathscr{W}^h_*(n)^t),$$

are the homology groups of a double complex

$$(C_s(O(1,n-1),\mathscr{W}^h_r(n)^t),d,\delta), \tag{2}$$

where C_* means the non-homogeneous chain complex of the homology of groups.

The first spectral sequence of (2) collapses from E^1, since $E^1_{p,q}=C_p(O(1,n-1),H_q(\mathscr{W}^h_*(n))^t)$, that is $E^1_{p,n-1}=C_p(O(1,n-1),\mathscr{S}t^h(n)^t)$ and $E^1_{p,q}=0$ for $q\neq n-1$. Then $E^2_{p,n-1}=E^\infty_{p,n-1}=H_p(O(1,n-1),\mathscr{S}t^h(n)^t)$, and

$$\mathbb{H}_i(O(1,n-1),\mathscr{W}^h_*(n)^t)\cong H_{i-n+1}(O(1,n-1),\mathscr{S}t^h(n)^t).$$

For the second spectral sequence of (2), we have

$$E^1_{p,q}=H_q(O(1,n-1),\mathscr{W}^h_p(n)^t).$$

The $O(1,n-1)$-module $\mathscr{W}^h_p(n)^t$ splits as a direct sum of induced modules along the orbits of $O(1,n-1)$, acting on the set of direct orthogonal decompositions, as in Section 4.1. Shapiro's lemma gives an isomorphism

$$H_q(O(1,n-1),\mathscr{W}^h_p(n)^t)$$
$$\cong\bigoplus_{q_0+q_1\cdots+q_p=q}^{n_0+n_1+\cdots+n_p=n}H_{q_0}(O(1,n_0-1),\mathbb{Q}^t)\otimes H_{q_1}(O(n_1),\mathbb{Q}^t)\otimes\cdots\otimes H_{q_p}(O(n_p),\mathbb{Q}^t).$$

Since the differential d^1 coincides with the differential of the bar complex $\mathscr{B}_*(\mathbf{M}^h,\mathbf{A})$, we get

$$E^2_{p,q}\cong H_{p,q}(\mathbf{A},\mathbf{M}^h)_n.$$

The last assertion of the theorem uses Corollary 3.3.2.

Proceeding in the same way, for the Euclidean case, leads to the result:

Theorem 4.2.2. *There is a spectral sequence*

$$E^2_{p,q}=H_{p,q}(\mathbf{A},\mathbf{M}^e)_n\Rightarrow H_{p+q-n+1}(E(1,n-1),\mathscr{S}t^e(n)^t),$$

with $E^2_{p,q}=0$ *for with* $E^2_{p,q}=0$ *for* $p\geqslant n/2$, $E^2_{n/2-1,q}=0$ *for* $n/2$ *even and* q *odd, or* $n/2$ *odd and* q *even.*

5. Calculations in low dimensions

We give applications in low dimensions of the previous spectral sequences.

5.1. Spherical setting

Example 5.1.1. The inspection of the spectral sequence of Theorem 4.1.1, for $n=2$, gives easily

Proposition 5.1.2. *There are isomorphisms*

$$H_q(O(2), \mathscr{S}t(2)^t) \cong H_{q+1}(O(2), \mathbb{Q}^t).$$

Example 5.1.3. (Well-known from the work of Dupont)
If $n = 4$, one gets isomorphisms

$$H_0(O(4), \mathbb{Q}^t) \cong H_1(O(4), \mathbb{Q}^t) = 0,$$

and the exact sequence

$$0 \to H_3(O(4), \mathbb{Q}^t) \to H_0(O(4), \mathscr{S}t(4)^t)$$
$$\to H_1(O(2), \mathbb{Q}^t) \otimes H_1(O(2), \mathbb{Q}^t) \to H_2(O(4), \mathbb{Q}^t) \to 0.$$

For $i > 0$ *even*, we have exact sequences

$$0 \to H_{i+3}(O(4), \mathbb{Q}^t) \to H_i(O(4), \mathscr{S}t(4)^t)$$
$$\to H_{i+2}(O(2) \times O(2), \mathbb{Q}^t) \to H_{i+2}(O(4), \mathbb{Q}^t) \to H_{i-1}(O(4), \mathscr{S}t(4)^t) \to 0$$

The proof is an easy consequence of Theorem 4.1.1 and the two remarks: $E^2_{p,q} = 0$, if $p \neq 1, 2$, and $E^2_{2,q} = 0$, if q is odd. In particular for i odd

$$H_i(O(4), \mathscr{S}t(4)^t) \cong H_{1,i+3}(\mathbf{A}, \mathbb{Q})_4.$$

Example 5.1.4. For $n = 6$, we note the following facts for the spectral sequence: $E^2_{p,q} = 0$, if $p \neq 1, 2, 3$. $E^1_{3,q} = 0$, if q is even; $E^2_{2,q} = E^\infty_{2,q}$; $E^2_{1,q} = E^\infty_{1,q}$, if q is odd. As a result, the graded pieces of $H_i(O(6), \mathscr{S}t(6)^t)$ are
For i *even*

$$E^\infty_{1,i+5} = E^2_{1,i+5} = H_{1,i+5}(\mathbf{A}, \mathbb{Q})_6,$$

$$E^\infty_{2,i+4} = E^2_{2,i+4} = H_{2,i+4}(\mathbf{A}, \mathbb{Q})_6,$$

$$E^\infty_{3,i+3} = E^3_{3,i+3} = \mathrm{Ker}\{d^2 : H_{3,i+3}(\mathbf{A}, \mathbb{Q})_6 \to H_{1,i+2}(\mathbf{A}, \mathbb{Q})_6\}.$$

For i *odd*, there are only two non-zero pieces

$$E^\infty_{1,i+5} = E^3_{1,i+5} = \mathrm{Coker}\{d^2 : H_{3,i+6}(\mathbf{A}, \mathbb{Q})_6 \to H_{1,i+5}(\mathbf{A}, \mathbb{Q})_6\},$$

$$E^\infty_{2,i+4} = E^2_{2,i+4} = H_{2,i+4}(\mathbf{A}, \mathbb{Q})_6.$$

If we take into account the weight decompositions, the results are more precise.
The graded pieces of $H_i(O(6), \mathscr{S}t^{(1)}(6)^t)$ are, for i *even*

$${}^{(1)}E^\infty_{1,i+5} = H_{1,i+5}(\mathbf{A}, \mathbb{Q})_6,$$

$$^{(1)}E^{\infty}_{2,i+4} = H^{(1)}_{2,i+4}(\mathbf{A}, \mathbb{Q})_6,$$

$$^{(1)}E^{\infty}_{3,i+3} = \mathrm{Ker}\{d^2 : H^{(1)}_{3,i+3}(\mathbf{A}, \mathbb{Q})_6 \to H_{1,i+2}(\mathbf{A}, \mathbb{Q})_6\}$$

and for i *odd*

$$^{(1)}E^{\infty}_{1,i+5} = \mathrm{Coker}\{d^2 : H^{(1)}_{3,i+6}(\mathbf{A}, \mathbb{Q})_6 \to H_{1,i+5}(\mathbf{A}, \mathbb{Q})_6\},$$

$$^{(1)}E^{\infty}_{2,i+4} = H^{(1)}_{2,i+4}(\mathbf{A}, \mathbb{Q})_6.$$

For i *even*
$H_i(O(6), \mathscr{S}t^{(2)}(6)^t) \cong H^{(2)}_{2,i+4}(\mathbf{A}, \mathbb{Q})_6 \oplus H^{(2)}_{3,i+3}(\mathbf{A}, \mathbb{Q})_6$.
$H_i(O(6), \mathscr{S}t^{(3)}(6)^t) = H^{(3)}_{3,i+3}(\mathbf{A}, \mathbb{Q})_6$.
For i *odd*
$H_i(O(6), \mathscr{S}t^{(2)}(6)^t) = H^{(2)}_{2,i+4}(\mathbf{A}, \mathbb{Q})_6$, and $H_i(O(6), \mathscr{S}t^{(3)}(6)^t) = 0$.

We end this section by noting a few results connected to the previous calculations. For $i \leqslant 3$ (see [3] Chapter 7, remark following 7.13)

$$H_i(O(4), \mathbb{Q}^t) \cong H_i(SU(2, \mathbb{C}), \mathbb{Q}),$$

this implies

$$H_0(O(4), \mathbb{Q}^t) = H_1(O(4), \mathbb{Q}^t) = 0.$$

On the other hand, for $i \leqslant 3$ (see [3] Corollary 9.18)

$$H_i(SU(2, \mathbb{C}), \mathbb{Q}) \cong H_i(SL(2, \mathbb{C}), \mathbb{Q})^+,$$

where the $+$ means the invariant part under complex conjugation. Finally, it is known that $H_2(SL(2, \mathbb{C}), \mathbb{Q})$ is the Milnor K-group $K_2(\mathbb{C})$, and $H_3(SL(2, \mathbb{C}), \mathbb{Q})$ is the indecomposable part of $K_3(\mathbb{C})$ modulo torsion, which is also called the Bloch group [21,20].

5.2. *Hyperbolic and Euclidean setting*

We apply the spectral sequences of Theorems 4.2.1 and 4.2.2.

Example 5.2.1. For $n = 2$, we have

Proposition 5.2.2. *There are isomorphisms*

(1) $H_q(O(1,1), \mathscr{S}t^h(2)^t) \cong H_{q+1}(O(1,1), \mathbb{Q}^t)$.
(2) $H_q(E(1,1), \mathscr{S}t^e(2)^t) \cong H_{q+1}(E(1,1), \mathbb{Q}^t)$.

Example 5.2.3. (See also Dupont [3])
For $n = 4$, $H_0(O(1,3), \mathbb{Q}^t) = H_1(O(1,3), \mathbb{Q}^t) = 0$, and the sequence

$$0 \to H_3(O(1,3), \mathbb{Q}^t) \to H_0(O(1,3), \mathscr{S}t^h(4)^t)$$
$$\to H_1(O(1,1), \mathbb{Q}^t) \otimes H_1(O(2), \mathbb{Q}^t) \to H_2(O(1,3), \mathbb{Q}^t) \to 0$$

is exact.

For $i>0$ *even*, the following sequences are exact

$$0 \to H_{i+3}(O(1,3),\mathbb{Q}^t) \to H_i(O(1,3),\mathscr{S}t^h(4)^t) \to H_{i+2}(O(1,1)\times O(2),\mathbb{Q}^t)$$

$$\to H_{i+2}(O(1,3),\mathbb{Q}^t) \to H_{i-1}(O(1,3),\mathscr{S}t^h(4)^t) \to 0.$$

In the Euclidean setting, $H_0(E(1,3),\mathbb{Q}^t)=H_1(E(1,3),\mathbb{Q}^t)=0$, and the sequence

$$0 \to H_3(E(1,3),\mathbb{Q}^t) \to H_0(E(1,3),\mathscr{S}t^e(4)^t)$$

$$\to H_1(E(1,1),\mathbb{Q}^t)\times H_1(O(2),\mathbb{Q}^t) \to H_2(E(1,3),\mathbb{Q}^t) \to 0$$

is exact.

For $i>0$ *even*, the following sequences are exact

$$0 \to H_{i+3}(E(1,3),\mathbb{Q}^t) \to H_i(E(1,3),\mathscr{S}t^e(4)^t) \to H_{i+2}(E(1,1)\times O(2),\mathbb{Q}^t)$$

$$\to H_{i+2}(E(1,3),\mathbb{Q}^t) \to H_{i-1}(E(1,3),\mathscr{S}t^e(4)^t) \to 0.$$

Example 5.2.4. For $n=6$, the graded pieces of $H_i(O(1,5),\mathscr{S}t^h(6)^t)$ and $H_i(E(1,5),\mathscr{S}t^e(6)^t)$ are as follows.

For i *even*

$$E^\infty_{0,i+5}=E^2_{0,i+5}=H_{0,i+5}(\mathbf{A},\mathbf{M}^\star)_6,$$

$$E^\infty_{1,i+4}=E^2_{1,i+4}=H_{2,i+4}(\mathbf{A},\mathbf{M}^\star)_6,$$

$$E^\infty_{2,i+3}=E^3_{2,i+3}=\mathrm{Ker}\{d^2: H_{2,i+3}(\mathbf{A},\mathbf{M}^\star)_6 \to H_{0,i+2}(\mathbf{A},\mathbf{M}^\star)_6\}.$$

For i *odd*, there are only two non-zero pieces

$$E^\infty_{0,i+5}=E^3_{0,i+5}=\mathrm{Coker}\{d^2: H_{2,i+6}(\mathbf{A},\mathbf{M}^\star)_6 \to H_{0,i+5}(\mathbf{A},\mathbf{M}^\star)_6\},$$

$$E^\infty_{1,i+4}=E^2_{1,i+4}=H_{1,i+4}(\mathbf{A},\mathbf{M}^\star)_6.$$

We note the following isomorphisms related to the previous calculations (see [3] Theorem 6.2 and Corollary 9.18)

$$H_i(O(1,3),\mathbb{Q}^t)\cong H_i(SL(2,\mathbb{C}),\mathbb{Q})^+,\quad \text{for } i\leqslant 3.$$

$$H_2(E(1,3),\mathbb{Q}^t)\cong H_1(SO(3),\mathbb{R}^3)\cong \Omega^1_{\mathbb{Q}}\mathbb{R}.$$

$$H_3(E(1,3),\mathbb{Q}^t)\cong H_2(SO(3),\mathbb{R}^3)\oplus H_0\left(SO(3),\bigwedge_{\mathbb{Q}}^3\mathbb{R}^3\right)\cong 0\oplus\mathbb{R},$$

where $\Omega^1_{\mathbb{Q}}\mathbb{R}$ is the space of Kähler differentials of $\mathbb{R}$, considered as a $\mathbb{Q}$-algebra. The last isomorphism is related to the Sydler-Jessen theorem [3].

6. The shuffle algebra of spherical Steinberg modules

We consider the direct sum

$$\mathbf{St} = \bigoplus_{n\geqslant 0} \mathcal{S}t(n),$$

with the already made convention $\mathcal{S}t(0)=\mathbb{Q}$. The description of Steinberg modules by direct orthogonal sum decompositions has the following interesting consequence: there is a natural graded commutative algebra structure on $\mathbf{St}$ induced by shuffles [11].

To be precise, let us introduce the direct sum, which we consider as a differential graded $\mathbb{Q}$-vector space

$$\mathbf{W} = \bigoplus_{n\geqslant 0, p\geqslant 0} \mathcal{W}_p(n).$$

One gets a structure of differential graded algebra on $\mathbf{W}$, from the shuffle products

$$\begin{aligned}
&\mathcal{W}_p(m) \times \mathcal{W}_q(n) \to \mathcal{W}_{p+q}(m+n),\\
&((E_1,\dots,E_p),(E_{p+1},\dots,E_{p+q}))\\
&\quad\mapsto \sum_{\sigma=(p,q)\text{-shuffle}} sign(\sigma)(E_{\sigma^{-1}(1)},\dots,E_{\sigma^{-1}(p+q)}),
\end{aligned}\tag{3}$$

where $sign(\sigma)$ is the signature of the shuffle σ, and a subspace of $\mathbb{R}^m$ (resp. $\mathbb{R}^n$) is identified to a subspace of $\mathbb{R}^{m+n}$ through the canonical map $\mathbb{R}^m \to \mathbb{R}^m \times 0$ (resp. $\mathbb{R}^n \to 0 \times \mathbb{R}^n$)). We remark that the compatibility of this product with the differential uses in an essential way the commutativity of the "assembly" $(U,V) \mapsto U \oplus V$. This structure induces a commutative algebra structure on $\mathbf{St}$, viewed as the homology of $\mathbf{W}$: we call this algebra the *Steinberg algebra*.

Proposition 6.0.5. *The weights in the Steinberg modules are compatible with the product of the Steinberg algebra. The vector space* $\mathbf{St}^{(1)} := \bigoplus_{n\geqslant 1} \mathcal{S}t^{(1)}(n)$, *is canonically isomorphic to the indecomposables of the Steinberg algebra.*

This result is a consequence of the relations between Euler idempotents and shuffle products, as stated in [9] Section 1.5 or [18] Section 2, which work in our situation.

We do not obtain a Hopf algebra structure at this level. In fact the coproduct, as a deconcatenation map [10,22], makes sense only on coinvariants of $\mathbf{St}$ under the action of orthogonal groups: this is the subject of the next section.

7. Hopf structure on homology with Steinberg modules coefficients

7.1. A Hopf algebra

We consider the graded $\mathbb{Q}$-vector space

$$\mathbf{HSt} = \bigoplus_{n\geqslant 0, q\geqslant 0} H_q(O(n), \mathcal{S}t(n)^t),$$

with component $\mathbb{Q}$ in degree $(0,0)$. The space **HSt** inherits the structure of a commutative algebra from the product of the Steinberg algebra but in fact we have more:

Proposition 7.1.1. *The algebra* **HSt** *has a structure of commutative Hopf algebra such that the subspace of coinvariants*

$$\mathbf{H}_0\mathbf{St} = \bigoplus_{n\geqslant 0} H_0(O(n), \mathscr{S}t(n)^t),$$

is a Hopf subalgebra.

To define the coproduct in reduced form, we start from a *deconcatenation map*, at the level of $\mathscr{W}_p(n)$

$$\mathscr{W}_p(n) \to \bigoplus_{V_1\oplus^\perp V_2=\mathbb{R}^n}^{p_1+p_2=p} \mathscr{W}_{p_1}(V_1)\otimes\mathscr{W}_{p_2}(V_2)$$

$$(E_1,\ldots,E_p)\mapsto \sum_{j=1}^{p-1}(E_1,\ldots,E_j)\otimes(E_{j+1},\ldots,E_p), \tag{4}$$

where the direct sum is extended to the set of direct orthogonal decompositions of $\mathbb{R}^n$ in two non-zero subspaces. This map induces a map of complexes

$$\mathscr{W}_*(n) \to \bigoplus_{V_1\oplus^\perp V_2=\mathbb{R}^n} \mathscr{W}_*(V_1)\otimes\mathscr{W}_*(V_2),$$

where the right-hand side is a direct sum of tensor products of complexes. Going to homology, we get a map

$$\mathscr{S}t(n) \to \bigoplus_{V_1\oplus^\perp V_2=\mathbb{R}^n} \mathscr{S}t(V_1)\otimes\mathscr{S}t(V_2).$$

Now by Shapiro's lemma and Künneth's formula, we arrive at the desired coproduct

$$\boldsymbol{\Delta}: \mathbf{HSt}\to\mathbf{HSt}\otimes\mathbf{HSt}$$

$$H_q(O(n),\mathscr{S}t(n)^t) \to H_q\left(O(n), \bigoplus_{V_1\oplus^\perp V_2=\mathbb{R}^n} \mathscr{S}t(V_1)^t\otimes\mathscr{S}t(V_2)^t\right)$$

$$\cong \bigoplus_{p_1+p_2=p,\ q_1+q_2=q,}^{n_1+n_2=n} H_{q_1}(O(n_1),\mathscr{S}t(n_1)^t)\otimes H_{q_2}(O(n_2),\mathscr{S}t(n_2)^t). \tag{5}$$

Observe that the previous Hopf algebra has a very natural *antipode*, given at the level of direct sum decompositions by

$$(E_1,E_2,\ldots,E_p)\mapsto(-1)^p(E_p,\ldots,E_2,E_1).$$

Let us define a *dimension shift* $\mathbf{HSt}[\bullet]$ by $\mathbf{HSt}[\bullet]_{n,q}=\mathbf{HSt}_{n,q-n}$. If we collect the spectral sequences of Theorem 4.1.1 for all dimensions, we get:

Proposition 7.1.2. *There is a spectral sequence Hopf algebras whose abutment is* $\mathbf{HSt}[\bullet]$ *and term* E^2 *is given by* $H_*(\mathbf{A}, \mathbb{Q})$.

Actually, what we mean is the following: recall the spectral sequence of Hopf algebras of Section 3.2

$$E^2_{p,q} = Tor^{\mathbf{A}}_{p,q}(\mathbb{Q}, \mathbb{Q}) \Rightarrow Tor^{\mathbf{C}_\Sigma}_{p+q}(\mathbb{Q}, \mathbb{Q}).$$

The E^2 term is the required one, and the abutment is $H_*(\mathscr{B}\mathbf{C}_\Sigma)$. Now the bar complex $\mathscr{B}\mathbf{C}_\Sigma$ is quasi-isomorphic to the total complex $Tot \oplus_n \Gamma_n$, where Γ_n is the bicomplex (1) in the proof of Theorem 4.1.1. This gives, as in the proof of Theorem 4.1.1, an isomorphism $H_*(\mathscr{B}\mathbf{C}_\Sigma) \cong \mathbf{HSt}[\bullet]$. It is easy to check that this isomorphism is in fact an isomorphism of Hopf algebras.

We have already remarked that, for $\mathbf{A}$ commutative, the bar construction is a Hopf algebra. By a theorem of Leray, such a Hopf algebra is a free commutative algebra on its indecomposable part. In particular, the Hochschild homology $H_*(\mathbf{A}, \mathbb{Q})$ is a commutative Hopf algebra, where the product is induced by a shuffle product. From the relation between shuffles and Euler idempotents *loc.cit.*, one finds that its indecomposable part is $H^{(1)}_*(\mathbf{A}, \mathbb{Q})$.

Analogously the indecomposables of the algebras $\mathbf{HSt}$ and $\mathbf{H}_0\mathbf{St}$ are respectively isomorphic to

$$\mathbf{HSt}^{(1)} := \bigoplus_{n,q} H_q(O(n), \mathscr{S}t^{(1)}(n)^t) \quad \text{and} \quad \mathbf{H}_0\mathbf{St}^{(1)} := \bigoplus_{n} H_0(O(n), \mathscr{S}t^{(1)}(n)^t).$$

7.2. Comodules

Let

$$\mathbf{HSt}^h = \bigoplus_{n\geqslant 0, q\geqslant 0} H_q(O(1, n-1), \mathscr{S}t^h(n)^t),$$

$$\mathbf{HSt}^e = \bigoplus_{n\geqslant 0, q\geqslant 0} H_q(E(1, n-1), \mathscr{S}t^e(n)^t),$$

and

$$\mathbf{H}_0\mathbf{St}^h = \bigoplus_{n\geqslant 0} H_0(O(1, n-1), \mathscr{S}t^h(n)^t), \qquad \mathbf{H}_0\mathbf{St}^e = \bigoplus_{n\geqslant 0} H_0(E(1, n-1), \mathscr{S}t^e(n)^t).$$

Proposition 7.2.1. *The spaces* $\mathbf{HSt}^h$ *and* $\mathbf{HSt}^e$ *are right comodules over the Hopf algebra* $\mathbf{HSt}$. *These structures induce on the subspaces* $\mathbf{H}_0\mathbf{St}^h$ *and* $\mathbf{H}_0\mathbf{St}^e$ *comodules structures over the Hopf subalgebra* $\mathbf{H}_0\mathbf{St}$.

For the proof, we just trace the origin of the coproducts

$$\Delta^h : \mathbf{HSt}^h \to \mathbf{HSt}^h \otimes \mathbf{HSt},$$

and

$$\Delta^e : \mathbf{HSt}^e \to \mathbf{HSt}^e \otimes \mathbf{HSt}.$$

They are induced by certain maps of complexes:

$$\mathscr{W}^h_*(n) \to \bigoplus_{V_1 \oplus^\perp V_2 = \mathbb{R}^n} \mathscr{W}_*(V_1^h) \otimes \mathscr{W}_*(V_2), \tag{6}$$

$$\mathscr{W}^e_*(n) \to \bigoplus_{V_1 \oplus V_2 = \mathbb{R}^n} \mathscr{W}_*(V_1^e) \otimes \mathscr{W}_*(V_2). \tag{7}$$

Here as usual the sum is extended, in the hyperbolic case, to the orthogonal decompositions by two non-zero subspaces, the first one being a geometric subspace. In the Euclidean case, it is extended to the direct sum decompositions by two non-zero subspaces, where the first one is a geometric subspace and $(V_1 \cap H) \oplus V_2$ is an orthogonal decomposition of the hyperplane $H = \{x_0 = 0\}$. The maps (6) and (7) are defined as deconcatenation maps

$$(E_0, \dots, E_p) \mapsto \sum_{j=0}^{p-1} (E_0, \dots, E_j) \otimes (E_{j+1}, \dots, E_p).$$

8. Connection with scissors congruence groups

We study the connection of the previous section with the scissors congruence groups of polytopes in classical geometries. In particular we are interested in a Hopf algebra of spherical polytopes, introduced by Sah [19, Chapter 6], more than 20 years ago. Actually the relation between Steinberg modules and scissors congruence groups had been discovered by Dupont [2] at about the same time.

8.1. Groups of polytopes

For more details on this section, we refer to [3]. We use the notations of [3, Chapter 2], for the chain complexes $\bar{C}_*(S^{n-1})$ and $C_*(S^{n-1})$ based on tuples of points of the unit sphere S^{n-1} in $\mathbb{R}^n$, except that we work up to torsion: precisely $\bar{C}_p(S^{n-1})$ is the $\mathbb{Q}$-vector space freely generated by the $(p+1)$-tuples $(a_0, \dots, a_p)$ of points of S^{n-1}, with the usual differential, and in the case of $C_p(S^{n-1})$, $(a_0, \dots, a_p)$ is moreover subjected to be contained in an hemisphere. Let $\bar{C}_*(S^{n-1})^{n-2}$ and $C_*(S^{n-1})^{n-2}$ be the corresponding subcomplexes generated by the tuples contained in $(n-2)$-dimensional subspheres.

The group of spherical polytopes $\mathbf{Pt}(S^{n-1})$, as considered by Sah, is isomorphic to $H_{n-1}(C_*(S^{n-1})/C_*(S^{n-1})^{n-2})$. One can also introduce a group of polytopes $\overline{\mathbf{Pt}}(S^{n-1})$ as $H_{n-1}(\bar{C}_*(S^{n-1})/\bar{C}_*(S^{n-1})^{n-2})$ (note that this last notation differs from the one in [3] Chapter 2). The *spherical scissors congruence* groups are then the groups of coinvariants $H_0(O(n), \mathbf{Pt}(S^{n-1})^t)$ and $H_0(O(n), \overline{\mathbf{Pt}}(S^{n-1})^t)$. We write these groups respectively $\mathscr{P}(S^{n-1})$ and $\bar{\mathscr{P}}(S^{n-1})$ (we put $\mathscr{P}(S^{n-1}) = \bar{\mathscr{P}}(S^{n-1}) = \mathbb{Q}$ for $n = 0$). In fact $\bar{\mathscr{P}}(S^{n-1})$ is 0 for n odd.

From [3] Theorem 7.4, since our polytope groups are $\mathbb{Q}$-vector spaces, we get isomorphisms

$$\mathscr{P}(S^{2m-1})/\Sigma\mathscr{P}(S^{2m-2}) \xrightarrow{\cong} \bar{\mathscr{P}}(S^{2m-1}), \tag{8}$$

where $\Sigma : \mathscr{P}(S^{n-2}) \to \mathscr{P}(S^{n-1})$ is the so-called suspension map, and

$$H_0(O(2m), \mathscr{S}t(2m)^t) \xrightarrow{\cong} \bar{\mathscr{P}}(S^{2m-1}). \tag{9}$$

Let us define the *orthonormalization* $\ll a_1, \dots, a_n \gg$ of a tuple $(a_1, \dots, a_n)$ of independent vectors of S^{2m-1} as the orthogonal direct sum decomposition of $\mathbb{R}^n$ associated to the flag $U_1 \subset U_2 \subset \cdots \subset U_{n-1}$, where U_l is the linear span of $\{a_1, \dots, a_l\}$. After [3] Remark following Theorem 3.5, the isomorphism (9) is induced by the maps

$$\begin{aligned} f_{2m} : \bar{C}_{2m-1}(S^{2m-1})/\bar{C}_{2m-1}(S^{2m-1})^{2m-2} &\to \mathscr{W}_{2m}(2m) \\ (a_1, \dots, a_{2m}) &\mapsto \sum_{\sigma} sign(\sigma) \ll a_{\sigma(1)}, \dots, a_{\sigma(2m)} \gg \end{aligned} \tag{10}$$

In hyperbolic and Euclidean geometry, the relation between scissors congruence groups and Steinberg modules is simpler. Denote respectively by H^{n-1} and E^{n-1} the hyperbolic space and the Euclidean affine space in dimension $n-1$. The *hyperbolic scissors congruence* group, written $\mathscr{P}(H^{n-1})$, is a $\mathbb{Q}$-vector space isomorphic to $H_0(O^+(1, n-1), \mathscr{S}t^h(n)^t)$. The *Euclidean scissors congruence* group $\mathscr{P}(E^{n-1})$ is isomorphic to $H_0(E(n-1), \mathscr{S}t^e(n)^t)$. In even dimension, the coinvariants of Steinberg modules under the actions of the groups $O(1, n-1)$ and $E(1, n-1)$ are zero. In odd dimension, we easily get

$$\mathscr{P}(H^{2m-1}) \cong H_0(O(1, 2m-1), \mathscr{S}t^h(2m)^t),$$

and

$$\mathscr{P}(E^{2m-1}) \cong H_0(E(1, 2m-1), \mathscr{S}t^e(2m)^t).$$

These isomorphisms have descriptions similar to the one given by the maps (10).

8.2. *The Sah algebra*

After Sah [19] Chapter 6, the direct sum

$$\mathbf{P}(S) = \bigoplus_{m \geqslant 0} \bar{\mathscr{P}}(S^{2m-1}),$$

has a natural structure of Hopf algebra, where the product is induced by the *join* of polytopes, and the coproduct is a combination of generalized *Dehn invariants*, the so-called *total Dehn invariant*. In this form the Sah algebra is the one introduced in [6], actually Sah considered a variant.

Theorem 8.2.1. *The Sah algebra* $\mathbf{P}(S)$ *is isomorphic to the Hopf algebra* $\mathbf{H}_0\mathbf{St}$.

The isomorphism stems from the Dupont maps (10). There are two parts in the proof: the first one relates the join to the shuffle product, and the second connects the total Dehn invariant to the deconcatenation map.

For $m=m_1+m_2$ we identify the sphere S^{2m_1-1} (resp. S^{2m_2-1}) to the unit sphere of $\mathbb{R}^{2m_1}\times 0$ (resp. $0\times\mathbb{R}^{2m_2}$) embedded in $\mathbb{R}^{2m}$.

Let $(a_1,\dots,a_{2m_1})$ (resp. $(b_1,\dots,b_{2m_2})$) be a simplex of $\bar{C}_{2m_1-1}(S^{2m_1-1})$ (resp. $\bar{C}_{2m_2-1}(S^{2m_2-1})$) and let $(A_1,\dots,A_{2m_1})$ (resp. $(B_1,\dots,B_{2m_2})$) be the orthonormalizations $\lll a_1,\dots,a_{2m_1}\ggg$ and $\lll b_1,\dots,b_{2m_2}\ggg$. The orthonormalization of the join $(a_1,\dots,a_{2m_1}, b_1,\dots,b_{2m_2})$ is exactly $(A_1,\dots,A_{2m_1},B_1,\dots,B_{2m_2})$. If we write $(c_1,\dots,c_{2m})$ for the simplex $(a_1,\dots,a_{2m_1},b_1,\dots,b_{2m_2})$ of S^{2m-1}, and $(C_1,\dots,C_{2m})$ for $(A_1,\dots,A_{2m_1},B_1,\dots,B_{2m_2})$, then for any $(2m_1,2m_2)$-shuffle τ, the orthonormalization in $\mathbb{R}^{2m}$ of $(c_{\tau^{-1}(1)},\dots,c_{\tau^{-1}(2m)})$ is exactly $(C_{\tau^{-1}(1)},\dots,C_{\tau^{-1}(2m)})$. This implies that the maps (10) commute with the join and the shuffle product, that is we have commutative diagrams

$$\begin{tikzcd}
\bar{C}_{2m_1-1}(S^{2m_1-1})\otimes\bar{C}_{2m_2-1}(S^{2m_2-1}) \arrow[r, "join"] \arrow[d, "f_{2m_1-1}\otimes f_{2m_2-1}"] & \bar{C}_{2m-1}(S^{2m-1}) \arrow[d, "f_{2m-1}"] \\
\mathscr{W}_{2m_1}(2m_1)\otimes\mathscr{W}_{2m_2}(2m_2) \arrow[r, "shuffle"] & \mathscr{W}_{2m}(2m)
\end{tikzcd}$$

This ends the first part of the proof.

For the second part, we want to precise the total Dehn invariant

$$\mathscr{D}_m:\bar{\mathscr{P}}(S^{2m-1})\to\bigoplus_{m_1+m_2=m}\bar{\mathscr{P}}(S^{2m_1-1})\otimes\bar{\mathscr{P}}(S^{2m_2-1}),$$

whose components are the generalized Dehn invariants. We first intend to make clear that $\mathscr{D}_m$ comes at the chain level from the map

$$\Phi_m:\bar{C}_{2m-1}(S^{2m-1})\to\bigoplus_{V_1\oplus^\perp V_2=\mathbb{R}^{2m}}\bar{C}_{2m_1-1}(S(V_1))\otimes\bar{C}_{2m_2-1}(S(V_2))$$
$$(a_1,\dots,a_{2m})\mapsto\sum_{\tau,m_1+m_2=m} sign(\tau)(a_{\tau(1)},\dots,a_{\tau(2m_1)})\otimes(b_{\tau(2m_1+1)},\dots,b_{\tau(2m)}),$$

where: $\dim V_1=2m_1$, $\dim V_2=2m_2$; the sum in the right-hand side is extended to the $(2m_1,2m_2)$-shuffles τ, with $m_1,m_2\neq 0$; and for $l>2m_1$, $b_{\tau(l)}$ is the normalized orthogonal projection of $a_{\tau(l)}$ onto the orthogonal subspace to the linear span of $\{a_{\tau(1)},\dots,a_{\tau(2m_1)}\}$. Actually Φ_m induces a map

$$\overline{\mathbf{Pt}}(S^{2m-1})\to\bigoplus_{V_1\oplus^\perp V_2=\mathbb{R}^{2m}}\overline{\mathbf{Pt}}(S(V_1))\otimes\overline{\mathbf{Pt}}(S(V_2)).$$

Then if one goes to the coinvariants under the actions twisted by the determinant, one can recognize the coproduct $\mathscr{D}_m$, as considered in [19, Chapter 6].

Now the maps Φ_m commute at the chain level with the maps f_n and the deconcatenation maps, that is we have commutative diagrams where $2m_1=\dim V_1$ and $2m_2=\dim V_2$

$$\begin{tikzcd}
\bar{C}_{2m-1}(S^{2m-1}) \arrow[r, "\Phi_m"] \arrow[d, "f_{2m}"] & \bigoplus_{V_1\oplus^\perp V_2=\mathbb{R}^{2m}}\bar{C}_{2m_1-1}(S(V_1))\otimes\bar{C}_{2m_2-1}(S(V_2)) \arrow[d, "\oplus f_{2m_1-1}\otimes f_{2m_2-1}"] \\
\mathscr{W}_{2m}(2m) \arrow[r, "\Delta"] & \bigoplus_{V_1\oplus^\perp V_2=\mathbb{R}^{2m}}\mathscr{W}_{2m_1}(V_1)\otimes\mathscr{W}_{2m_2}(V_2)
\end{tikzcd}$$

This completes the proof.

After Sah [19], the direct sums $\mathbf{P}(H) = \oplus_m \mathscr{P}(H^{2m-1})$ and $\mathbf{P}(E) = \oplus_m \mathscr{P}(E^{2m-1})$ are comodules over the Sah algebra, with a coproduct given by generalized Dehn invariants.

Theorem 8.2.2. *Through the previous isomorphisms between groups of polytopes and Steinberg modules, the comodules* $\mathbf{P}(H)$ *and* $\mathbf{P}(E)$ *become respectively the comodules* $\mathbf{H}_0\mathbf{St}^h$ *and* $\mathbf{H}_0\mathbf{St}^e$, *over* $\mathbf{H}_0\mathbf{St}$.

It is easy to adapt the proof of Theorem 8.2.1.

Question 8.2.3. Is there also a concrete interpretation for the whole algebra **HSt**? This would be useful to explicit new scissors invariants of polytopes, different from volumes and generalized Dehn invariants.

Question 8.2.4. By the results of Section 2.4, each scissors congruence group $\bar{\mathscr{P}}(S^{2m-1})$ is graded by the weights of the Steinberg module $\mathscr{S}t(2m)$, with non-zero pieces $\bar{\mathscr{P}}^{(r)}(S^{2m-1})$ only for $r = 1, \ldots, m$. On the other hand, each piece $\bar{\mathscr{P}}^{(r)}(S^{2m-1})$ is filtered by the spectral sequences of Theorem 4.1.2, and the graded space of this filtration has at most $m - r + 1$ non-zero pieces. So a natural question is: what is the geometric meaning of this structure? It results from the Proposition 6.0.5 that the weights are related to the indecomposables of the Sah algebra. The interpretation of the filtration on the weight pieces does not seem so clear. In particular what is the connection between this filtration and the filtration by the primitives? See also Remark 10.1.5.

9. Cobar constructions

9.1. The cobar construction

The normalized cobar construction goes from the category of supplemented differential coalgebras and comodules to the category of differential algebras and modules. For a differential coalgebra B and a comodule N, we write $\mathscr{C}B$ and $\mathscr{C}(N,B)$ for the corresponding differential algebra and module. Recall the cobar construction over a graded differential supplemented $\mathbb{Q}$-coalgebra $(B = \bar{B} \oplus \mathbb{Q}, \beta)$. As an algebra $\mathscr{C}B$ is the graded tensor algebra $\mathbf{T}\mathrm{s}^{-1}\bar{B}$ over the desuspension of $\bar{B}$, with the concatenation product. The differential $D = \partial + \beta$ is defined by

$$\partial(\mathrm{s}^{-1}b_1 \otimes \cdots \otimes \mathrm{s}^{-1}b_r) = \sum_{i=1}^{i=r} \varepsilon_i \sum_j (-1)^{\deg b'_{ij}+1} \mathrm{s}^{-1}b_1 \otimes \cdots \otimes \mathrm{s}^{-1}b'_{ij} \otimes \mathrm{s}^{-1}b''_{ij} \otimes \cdots \otimes \mathrm{s}^{-1}b_r,$$

where $\sum_j b'_{ij} \otimes b''_{ij} = \bar{\Delta} b_i$, $\varepsilon_i = (-1)^{\sum_{j=1}^{i-1} \deg \mathrm{s}^{-1}b_j}$, and

$$\beta(b_1 \otimes \cdots \otimes b_r) = -\sum_{i=1}^{i=r} \varepsilon_i \mathrm{s}^{-1}b_1 \otimes \cdots \otimes \mathrm{s}^{-1}\beta(b_i) \otimes \cdots \otimes \mathrm{s}^{-1}b_r.$$

We refer to [5] for the formulae in case of the presence of a comodule. The cobar complex can be considered as the total complex of a bicomplex with the two differentials ∂, β of degree -1. The element $\mathrm{s}^{-1}b_1 \otimes \cdots \otimes \mathrm{s}^{-1}b_r$ has total degree $\varepsilon = \sum_{j=1}^{r} \deg \mathrm{s}^{-1}b_j$ and bidegree $(-r, \varepsilon + r)$. This bicomplex leads to the classical Eilenberg–Moore spectral sequence [4,12].

We think of the homology of $(\mathscr{C}B, D)$ and $(\mathscr{C}(N,B), D)$ as the Hochschild cohomology of B, denoted by $H^l(B,\mathbb{Q})$ and $H^l(B,N)$. For example **HSt** is considered as a graded coalgebra, with zero differential. Its cohomology groups are the homology groups of a (q,n)-bigraded *reduced* cobar complex with pieces

$$\mathscr{C}_{-p,q}(\mathbf{HSt})_n = \bigoplus_{q_1+\cdots+q_p=q}^{n_1+\cdots+n_p=n} H_{q_1}(O(n_1), \mathscr{S}t(n_1)^t) \otimes \cdots \otimes H_{q_p}(O(n_p), \mathscr{S}t(n_p)^t),$$

where the n_j are non-zero even integers and the differential comes from the coproduct in the standard way. In what follows, we write $H_q^p(\mathbf{HSt},\mathbb{Q})_n$ the pieces of these cohomology groups, where n is the dimension and q the total homological degree relative to the orthogonal group.

We will also consider the cohomology of **HSt** with coefficients in the comodules $\mathbf{HSt}^h$ and $\mathbf{HSt}^e$. This cohomology is given by the homology groups of cobar complexes, with pieces

$$\mathscr{C}_{-p,q}(\mathbf{HSt}^h, \mathbf{HSt})_n =$$

$$\bigoplus_{q_0+\cdots+q_p=q}^{n_0+\cdots+n_p=n} H_{q_0}(O(1, n_0-1), \mathscr{S}t^h(n_0)^t) \otimes H_{q_1}(O(n_1), \mathscr{S}t(n_1)^t) \otimes \cdots \otimes H_{q_p}(O(n_p), \mathscr{S}t(n_p)^t),$$

$$\mathscr{C}_{-p,q}(\mathbf{HSt}^e, \mathbf{HSt})_n =$$

$$\bigoplus_{q_0+\cdots+q_p=q}^{n_0+\cdots+n_p=n} H_{q_0}(E(1, n_0-1), \mathscr{S}t^e(n_0)^t) \otimes H_{q_1}(O(n_1), \mathscr{S}t(n_1)^t) \otimes \cdots \otimes H_{q_p}(O(n_p), \mathscr{S}t(n_p)^t).$$

In what follows, we write $H_q^p(\mathbf{HSt}, \mathbf{HSt}^\star)_n$, for the pieces of the cohomology group $H^p(\mathbf{HSt}, \mathbf{HSt}^\star)$.

The following result of Moore on the cobar–bar construction ([5] Prop. 2.14) is crucial for us:

Proposition 9.1.1. *Let A be a differential algebra and M be a differential A-module, there are morphisms of differential algebras: $\mathscr{C}\mathscr{B}A \to A$, and differential modules: $\mathscr{C}(\mathscr{B}(M,A),\mathscr{B}A) \to M$, coming from the adjoint pair of functors $(\mathscr{C},\mathscr{B})$, which are quasi-isomorphisms.*

In what follows, we are interested in the cobar–bar constructions relative to the algebra **A** and the **A**-modules $\mathbf{M}^h$ and $\mathbf{M}^e$. From the previous proposition, the homology of the differential algebra $\mathscr{C}\mathscr{B}\mathbf{A}$ satisfies $H_*\mathscr{C}\mathscr{B}\mathbf{A}=\mathbf{A}$, which means that for each n, $H_q([\mathscr{C}\mathscr{B}\mathbf{A}]_n)=H_q(O(n),\mathbb{Q}^t)$, where $[\mathscr{C}\mathscr{B}\mathbf{A}]_n$ is the subcomplex relative to the dimension n. Similarly, $H_*\mathscr{C}(\mathscr{B}(\mathbf{M}^\star,\mathbf{A}),\mathscr{B}\mathbf{A})=\mathbf{M}^\star$. And the homology in dimension n are respectively given by $H_q(O(1,n-1),\mathbb{Q}^t)$ and $H_q(E(1,n-1),\mathbb{Q}^t)$. Remark also that the differential algebras $\mathscr{C}\mathscr{B}\mathbf{C}$ and $\mathscr{C}\mathscr{B}\mathbf{C}_\Sigma$ are quasi isomorphic to $\mathbf{A}$.

In our context, these cobar–bar constructions will appear from the consideration of certain "pre-cobar" complexes which we now introduce.

9.2. Pre-cobar bicomplexes

To a Euclidean vector space E, we associate a bicomplex of $O(E)$-modules

$$\Omega\mathscr{W}(E)=(\Omega\mathscr{W}_{*,*}(E),\partial,d), \tag{11}$$

where the differentials have degree respectively $(-1,0)$ and $(0,-1)$. It is obtained from the complex $\mathscr{W}_*(E)$ by a "pre-cobar" construction in the following manner

$$\Omega\mathscr{W}_{p,-s}(E)=\bigoplus_{V_1\oplus^\perp\cdots\oplus^\perp V_s=E}^{p_1+\cdots+p_s=p}\mathscr{W}_{p_1}(V_1)\otimes\cdots\otimes\mathscr{W}_{p_s}(V_s),$$

where the V_i are non-zero subspaces of E. The two differentials d and ∂ come respectively from the differential of $\mathscr{W}_*(E)$ and the deconcatenation maps (4) of Section 7

$$\mathscr{W}_*(V)\to\bigoplus_{V_1\oplus^\perp V_2=V}\mathscr{W}_*(V_1)\otimes\mathscr{W}_*(V_2).$$

Similarly in the hyperbolic and Euclidean setting, we can define bicomplexes

$$\Omega\mathscr{W}(E^h)=(\Omega\mathscr{W}_{*,*}(E^h),\partial,d), \tag{12}$$

and

$$\Omega\mathscr{W}(E^e)=(\Omega\mathscr{W}_{*,*}(E^e),\partial,d), \tag{13}$$

with differentials of degree $(-1,0)$ and $(0,-1)$, which are respectively bicomplexes of $O(E^h)$ and $O(E^e)$-modules. They are given by

$$\Omega\mathscr{W}_{p,-s}(E^h)=\bigoplus_{V_0\oplus^\perp\cdots\oplus^\perp V_s=E}^{p_0+\cdots+p_s=p}\mathscr{W}_{p_0}(V_0^h)\otimes\mathscr{W}_{p_1}(V_1)\otimes\cdots\otimes\mathscr{W}_{p_s}(V_s),$$

where the V_i are non-zero subspaces of E, and

$$\Omega\mathscr{W}_{p,-s}(E^e)=\bigoplus_{V_0\oplus\cdots\oplus V_s=E}^{p_0+\cdots+p_s=p}\mathscr{W}_{p_0}(V_0^e)\otimes\mathscr{W}_{p_1}(V_1)\otimes\cdots\otimes\mathscr{W}_{p_s}(V_s),$$

where as before the sum is extended to direct sum decompositions $V_0\oplus\cdots\oplus V_s$ by non-zero subspaces, such that the direct sum $(V_0\cap H)\oplus V_1\ldots\oplus V_s$ is orthogonal. The differentials d and ∂ come respectively from the differential of $\mathscr{W}_*(E^\star)$ and deconcatenation maps.

9.3. Hyperhomology of the precobar complex: the spherical case

If E is the canonical Euclidean space $\mathbb{R}^n$, we write $\Omega\mathscr{W}(n)$ for $\Omega\mathscr{W}(E)$. Let $(Tot_*\Omega\mathscr{W}(n), \partial \pm d)$ be the total complex of $(\Omega\mathscr{W}_{*,*}(n), \partial, d)$. This complex is a complex of $O(n)$-modules for the natural action twisted by the determinant.

Theorem 9.3.1. *The hyperhomology groups $\mathbb{H}_*(O(n), Tot_*\Omega\mathscr{W}(n)^t)$ are isomorphic to the homology groups $H_*(O(n), \mathbb{Q}^t)$.*

Consider the hyperhomology spectral sequence, converging to $\mathbb{H}_*(O(n), Tot_*\Omega\mathscr{W}(n)^t)$, relative to the filtration by v of the bicomplex

$$(C_\gamma(O(n), Tot_v\Omega\mathscr{W}(n)^t), \partial \pm d, \delta),$$

We will prove that it collapses from E^2. The crucial point is that the graded complex E^1 with differential d^1 identifies to the cobar–bar complex $[\mathscr{C}\mathscr{B}\mathbf{A}]_n$. As a result

$$E^2_{p,q} = 0, \quad \text{for } p \neq 0,$$

$$E^2_{0,q} = H_q([\mathscr{C}\mathscr{B}\mathbf{A}]_n) = \mathbf{A}_{n,q} = H_q(O(n), \mathbb{Q}^t).$$

To justify the previous assertion, we will just identify the deconcatenation map of the cobar construction as the map $d^1 : E^1_{*,0} \to E^1_{*,1}$. This last map is the map

$$H_q(O(n), \mathscr{W}_p(n)^t) \to H_q\left(O(n), \bigoplus_{V_1 \oplus^\perp V_2 = \mathbb{R}^n}^{p_1+p_2=p} \mathscr{W}_{p_1}(V_1)^t \otimes \mathscr{W}_{p_2}(V_2)^t\right),$$

induced by the map (4) of Section 7. From Shapiro's lemma and Künneth's formula, we get the map

$$H_q(O(n), \mathscr{W}_p(n)^t)$$

$$\to \bigoplus_{p_1+p_2=p,\ q_1+q_2=q,}^{n_1+n_2=n} H_{q_1}(O(n_1), \mathscr{W}_{p_1}(n_1)^t) \otimes H_{q_2}(O(n_2), \mathscr{W}_{p_2}(n_2)^t).$$

Through the isomorphisms of Section 4

$$H_i(O(l), \mathscr{W}_j(l)^t) \cong \bigoplus_{l_1+\cdots+l_j=l}^{i_1+\cdots+i_j=i} H_{i_1}(O(l_1), \mathbb{Q}^t) \otimes \cdots \otimes H_{i_j}(O(l_j), \mathbb{Q}^t),$$

we recognize the classical deconcatenation coproduct of the tensor algebra over $\bar{\mathbf{A}}$ placed in degree -1. It is easy to complete the proof.

The previous hyperhomology can be viewed as the homology of the total complex of the tricomplex

$$(C_\gamma(O(n), \Omega\mathscr{W}_{\beta,\alpha}(n)^t), \partial, d, \delta). \tag{14}$$

We consider another bicomplex deduced from this one

$$\left(\left(\bigoplus_{\alpha+\gamma=a} C_\gamma(O(n), \Omega\mathscr{W}_{b,\alpha}(n)^t)\right)_{a,b}, \partial \pm \delta, d\right).$$

The second spectral sequence of this bicomplex collapses. In fact, from Salomon–Tits, $E^1_{p,q} = 0$, for $p \neq n$, and $(E^1_{n,*}, d^1)$ coincides with the total complex of a bicomplex

$$(C_\gamma(O(n), \Omega\mathscr{S}t_v(n)^t), \partial, \delta), \tag{15}$$

where

$$\Omega\mathscr{S}t_v(n) = \bigoplus_{V_1 \oplus^\perp \cdots \oplus^\perp V_{-v} = \mathbb{R}^n} \mathscr{S}t(V_1) \otimes \cdots \otimes \mathscr{S}t(V_{-v}),$$

with obvious differentials ∂ and δ. We arrive at the result:

Theorem 9.3.2. *The hyperhomology groups $\mathbb{H}_*(O(n), \Omega\mathscr{S}t_*(n)^t)$ are isomorphic to the homology groups $H_{*+n}(O(n), \mathbb{Q}^t)$.*

If we go back to the bicomplex (15) and apply to the hyperhomology spectral of $\mathbb{H}_*(O(n), \Omega\mathscr{S}t_*(n)^t)$ the usual Shapiro-Künneth process, we get:

Corollary 9.3.3. *There is a spectral sequence*

$$E^2_{-p,q} = H^p_q(\mathbf{HSt}, \mathbb{Q})_n \Rightarrow H_{-p+q+n}(O(n), \mathbb{Q}^t).$$

A closer look at this spectral sequence for $n = 2m$, taking into account the results of Proposition 3.2.1, shows that

$$\mathbb{H}_{-m}(O(2m), \Omega\mathscr{S}t_*(2m)^t) = E^\infty_{-m,0} = E^2_{-m,0} \cong H_m(O(2m), \mathbb{Q}^t), \tag{16}$$

and

$$\mathbb{H}_{-m+1}(O(2m), \Omega\mathscr{S}t_*(2m)^t) = E^\infty_{-m+1,0} = E^2_{-m+1,0} \cong H_{m+1}(O(2m), \mathbb{Q}^t). \tag{17}$$

9.4. Hyperhomology of the precobar complex: the hyperbolic and Euclidean cases

We now resume the previous results in the hyperbolic and Euclidean setting. We write $\Omega\mathscr{W}^\star_{*,*}(n)$ for the bicomplex $\Omega\mathscr{W}_{*,*}(E^\star)$, when $E = \mathbb{R}^n$. The total complexes $Tot_*\Omega\mathscr{W}^h(n)$ and $Tot_*\Omega\mathscr{W}^e(n)$ are respectively complexes of modules over $O(1, n-1)$ and $E(1, n-1)$, for the natural actions twisted by the determinant.

Theorem 9.4.1. *The hyperhomology groups $\mathbb{H}_*(O(1, n-1), Tot_*\Omega\mathscr{W}^h(n)^t)$ are isomorphic to the homology groups $H_*(O(1, n-1), \mathbb{Q}^t)$.*

The hyperhomology groups $\mathbb{H}_*(E(1, n-1), Tot_*\Omega\mathscr{W}^e(n)^t)$ are isomorphic to the homology groups $H_*(E(1, n-1), \mathbb{Q}^t)$.

The proof follows the same lines as the one of Theorem 9.3.1.

Now we consider the tricomplex

$$(C_\gamma(O(1, n-1), \Omega\mathscr{W}^h_{\beta,\alpha}(n)^t, \partial, d, \delta). \tag{18}$$

The second spectral sequence of the associated bicomplex

$$\left(\left(\bigoplus_{\alpha+\gamma=a} C_\gamma(O(1,n-1),\Omega\mathscr{W}^h_{b,\alpha}(n)^t)\right)_{a,b}, \partial \pm \delta, d\right),$$

collapses. In fact, from Salomon–Tits, $E^1_{p,q}=0$, for $p \neq n-1$, and $(E^1_{n-1,*}, d^1)$ coincides with the total complex of a bicomplex

$$(C_\gamma(O(1,n-1),\Omega\mathscr{S}t^h_v(n)^t),\partial,\delta),$$

where

$$\Omega\mathscr{S}t^h_v(n) = \bigoplus_{V_0\oplus^\perp V_1\oplus^\perp\cdots\oplus^\perp V_{-v}=\mathbb{R}^n} \mathscr{S}t(V^h_0)\otimes\mathscr{S}t(V_1)\otimes\cdots\otimes\mathscr{S}t(V_{-v})$$

with obvious differentials.

We have similar observations in the Euclidean setting, concerning the following multicomplexes

$$(C_\gamma(E(1,n-1),\Omega\mathscr{W}^e_{\beta,\alpha}(n)^t),\partial,d,\delta), \tag{19}$$

and

$$(C_\gamma(E(1,n-1),\Omega\mathscr{S}t^e_v(n)^t),\partial,\delta).$$

This leads to:

Theorem 9.4.2. *The hyperhomology groups* $\mathbb{H}_*(O(1,n-1),\Omega\mathscr{S}t^h_*(n)^t)$ *are isomorphic to the homology groups* $H_{*+n-1}(O(1,n-1),\mathbb{Q}^t)$.

The hyperhomology groups $\mathbb{H}_*(E(1,n-1),\Omega\mathscr{S}t^e_*(n)^t)$ *are isomorphic to the homology groups* $H_{*+n-1}(E(1,n-1),\mathbb{Q}^t)$.

As a corollary:

Corollary 9.4.3. *There are spectral sequences*

$$E^2_{-p,q} = H^p_q(\mathbf{HSt},\mathbf{HSt}^h)_n \Rightarrow H_{-p+q+n-1}(O(1,n-1),\mathbb{Q}^t).$$

$$E^2_{-p,q} = H^p_q(\mathbf{HSt},\mathbf{HSt}^e)_n \Rightarrow H_{-p+q+n-1}(E(1,n-1),\mathbb{Q}^t).$$

Here again, as in the spherical case, a closer inspection of these spectral sequences for $n=2m$, taking into account the results of Proposition 3.3.1, yields the isomorphisms

$$\begin{aligned}\mathbb{H}_{-m+1}(O(1,2m-1),\Omega\mathscr{S}t^h_*(2m)^t) &= E^\infty_{-m+1,0} = E^2_{-m+1,0}\\ &\cong H_m(O(1,2m-1),\mathbb{Q}^t),\\ \mathbb{H}_{-m+1}(E(1,2m-1),\Omega\mathscr{S}t^e_*(2m)^t) &= E^\infty_{-m+1,0} = E^2_{-m+1,0}\\ &\cong H_m(E(1,2m-1),\mathbb{Q}^t),\end{aligned} \tag{20}$$

and

$$\begin{aligned}\mathbb{H}_{-m+2}(O(1,2m-1),\Omega\mathscr{S}t_*^h(2m)^t) &= E^{\infty}_{-m+2,0} = E^2_{-m+2,0}\\ &\cong H_{m+1}(O(1,2m-1),\mathbb{Q}^t),\\ \mathbb{H}_{-m+2}(E(1,2m-1),\Omega\mathscr{S}t_*^e(2m)^t) &= E^{\infty}_{-m+2,0} = E^2_{-m+2,0}\\ &\cong H_{m+1}(E(1,2m-1),\mathbb{Q}^t).\end{aligned} \tag{21}$$

10. Cohomology of Dehn complexes

10.1. Dehn complexes for spherical polytopes

We define the m-th Dehn complex as the $2m$-graded part of the reduced cobar complex of the Sah algebra. This complex was introduced by Goncharov [6] and it looks like

$$\mathscr{D}(m) : \bar{\mathscr{P}}(S^{2m-1}) \to \bigoplus_{m_1+m_2=m} \bar{\mathscr{P}}(S^{2m_1-1}) \otimes \bar{\mathscr{P}}(S^{2m_2-1}) \to \cdots$$

$$\to \bigoplus_{m_1+\cdots+m_s=m} \bar{\mathscr{P}}(S^{2m_1-1}) \otimes \cdots \otimes \bar{\mathscr{P}}(S^{2m_s-1}) \to \cdots \to \otimes^m \bar{\mathscr{P}}(S^1),$$

where $\bar{\mathscr{P}}(S^{2m-1})$ is in cohomological degree one and the m_i are non-zero integers. The cohomology group $H^1(\mathscr{D}(m))$ is the component in dimension $2m$ of the primitive part of $\mathbf{H}_0\mathbf{St}$.

Theorem 10.1.1. *The cohomology of the Dehn complex $\mathscr{D}(m)$, in degrees m and $m-1$, is given by*

$$H^m(\mathscr{D}(m)) \cong H_m(O(2m),\mathbb{Q}^t).$$

$$H^{m-1}(\mathscr{D}(m)) \cong H_{m+1}(O(2m),\mathbb{Q}^t).$$

The proof is a consequence of the isomorphisms (16) and (17) of Section 9.3, if we identify the Dehn complexes to the graded parts of the cobar complex of the algebra $\mathbf{H}_0\mathbf{St}$. This cobar complex appears as the complex $(E^1_{*,0}, d^1)$ of the hyperhomology spectral sequence in Corollary 9.3.3.

We retrieve the known cohomology of the Dehn complex for S^3: this particular case was also implicit in Example 5.1.3.

Since $\bar{\mathscr{P}}(S^1) \cong H_0(O(2),\mathscr{S}t(2)^t) \cong H_1(O(2),\mathbb{Q}^t)$, the following proposition is not a surprise.

Proposition 10.1.2. *The natural map*

$$\bigotimes^m H_1(O(2),\mathbb{Q}^t) \to H_m(O(2m),\mathbb{Q}^t)$$

is surjective.

From Theorem 9.3.1 we know that, for the tricomplex

$$\Xi = (C_\gamma(O(2m), \Omega\mathscr{W}_{\beta,\alpha}(2m)^t))_{\alpha,\beta,\gamma}$$

of Section 9, there is an isomorphism

$$H_m(O(2m), \mathbb{Q}^t) \cong H_m(Tot\Xi).$$

Resuming the proof of Theorem 9.3.1 for the sub-tricomplex $\Xi_{\gamma\leqslant -m}$ of Ξ, we get easily an isomorphism

$$\overset{m}{\otimes} H_1(O(2), \mathbb{Q}^t) \cong H_m(Tot(\Xi_{\gamma\leqslant -m})).$$

One can see also, from the spectral sequence used in the proof of Theorem 9.3.2, that there is an isomorphism

$$H_m(Tot(\Xi_{\gamma\leqslant -m})) \cong \overset{m}{\otimes} H_0(O(2), \mathscr{S}t(2)^t).$$

We observe now that the inclusion $\Xi_{\gamma\leqslant -m} \subset \Xi$, induces a surjective map: $H_m(Tot(\Xi_{\gamma\leqslant -m})) \to H_m(Tot\Xi)$, which actually corresponds to the surjection

$$\overset{m}{\otimes} \bar{\mathscr{P}}(S^1) \to H^m(\mathscr{D}(m)).$$

So the proof will be completed if we show that the map

$$\overset{m}{\otimes} H_1(O(2), \mathbb{Q}^t) \to H_m(O(2m), \mathbb{Q}^t),$$

induced by the last surjection, is the natural map up to sign. This follows from the lemma:

Lemma 10.1.3. *Let a_i, $i = 1, \dots, m$, be elements in $H_1(O(2), \mathbb{Q}^t)$ and $a_1 \dots a_m$ their product in $H_m(O(2m), \mathbb{Q}^t)$, the elements of degree m in $\mathscr{C}\mathscr{B}\mathbf{A}$: $\mathrm{s}^{-1}[\mathrm{s}(a_1 \dots a_m)]$ and $(-1)^m \mathrm{s}^{-1}[\mathrm{s}a_1] \otimes \cdots \otimes \mathrm{s}^{-1}[\mathrm{s}a_m]$, are two homologous cycles.*

Proof. In the cobar–bar complex of $\mathbf{A}$, we have

$$\begin{aligned} &D(\mathrm{s}^{-1}[\mathrm{s}(a_1 \cdots a_{i-1})|\mathrm{s}a_i] \otimes \mathrm{s}^{-1}[\mathrm{s}a_{i+1}] \otimes \cdots \otimes \mathrm{s}^{-1}[\mathrm{s}a_m]) \\ &\quad = -\mathrm{s}^{-1}[\mathrm{s}(a_1 \cdots a_i)] \otimes \mathrm{s}^{-1}[\mathrm{s}a_{i+1}] \otimes \cdots \otimes \mathrm{s}^{-1}[\mathrm{s}a_m] \\ &\qquad + \mathrm{s}^{-1}[\mathrm{s}(a_1 \cdots a_{i-1})] \otimes \mathrm{s}^{-1}[\mathrm{s}a_i] \otimes \mathrm{s}^{-1}[\mathrm{s}a_{i+1}] \otimes \cdots \otimes \mathrm{s}^{-1}[\mathrm{s}a_m]. \end{aligned}$$

It follows that

$$\begin{aligned} &D(\mathrm{s}^{-1}[\mathrm{s}(a_1 \cdots a_{m-1})|\mathrm{s}a_m] - \mathrm{s}^{-1}[\mathrm{s}(a_1 \cdots a_{m-2})|\mathrm{s}a_{m-1}] \otimes \mathrm{s}^{-1}[\mathrm{s}a_m] + \cdots \\ &\quad + (-1)^m \mathrm{s}^{-1}[\mathrm{s}a_1|\mathrm{s}a_2] \otimes \mathrm{s}^{-1}[\mathrm{s}a_3] \otimes \cdots \otimes \mathrm{s}^{-1}[\mathrm{s}a_m]) \\ &\qquad = (-1)^m \mathrm{s}^{-1}[\mathrm{s}a_1] \otimes \cdots \otimes \mathrm{s}^{-1}[\mathrm{s}a_m] - \mathrm{s}^{-1}[\mathrm{s}(a_1 \dots a_m)]. \qquad \square \end{aligned}$$

For $m = 3$, we can push the results a little bit further:

Theorem 10.1.4. *The cohomology of the Dehn complex* $\mathscr{D}(3)$ *relative to* S^5 *reads*

$$H^1(\mathscr{D}(3)) \cong H_{1,5}(\mathbf{A},\mathbb{Q})_6, \quad H^2(\mathscr{D}(3)) \cong H_4(O(6),\mathbb{Q}^t),$$
$$H^3(\mathscr{D}(3)) \cong H_3(O(6),\mathbb{Q}^t).$$

Note that $H_{1,5}(\mathbf{A},\mathbb{Q})_6$ is a quotient of $H_5(O(6),\mathbb{Q}^t)$.

For the proof that $H^1(\mathscr{D}(3)) \cong H_{1,5}(\mathbf{A},\mathbb{Q})_6$, we have to look at the kernel of the coproduct

$$H_0(O(6),\mathscr{S}t(6)^t) \to (H_0(O(2),\mathscr{S}t(2)^t) \otimes H_0(O(4),\mathscr{S}t(4)^t)) \oplus (H_0(O(4),\mathscr{S}t(4)^t) \otimes H_0(O(2),\mathscr{S}t(2)^t)).$$

From Sections 4 and 9, we know that

$$H_0(O(6),\mathscr{S}t(6)^t) \cong \mathbb{H}_6(O(6),\mathscr{W}_*(6)^t),$$

and

$$\begin{aligned}
&(H_0(O(2),\mathscr{S}t(2)^t) \otimes H_0(O(4),\mathscr{S}t(4)^t))\\
&\quad \oplus (H_0(O(4),\mathscr{S}t(4)^t) \otimes H_0(O(2),\mathscr{S}t(2)^t))\\
&\quad\quad \cong (\mathbb{H}_2(O(2),\mathscr{W}_*(2)^t) \otimes \mathbb{H}_4(O(4),\mathscr{W}_*(4)^t))\\
&\quad\quad\quad \oplus (\mathbb{H}_4(O(4),\mathscr{W}_*(4)^t) \otimes \mathbb{H}_2(O(2),\mathscr{W}_*(2)^t))\\
&\quad\quad \cong \mathbb{H}_6\left(O(6), \bigoplus_{V_1 \oplus^\perp V_2 = \mathbb{R}^6} \mathscr{W}_*(V_1) \otimes \mathscr{W}_*(V_2)\right).
\end{aligned}$$

Through these isomorphisms, the coproduct is induced by the deconcatenation map

$$\mathscr{W}_*(6) \to \bigoplus_{V_1 \oplus^\perp V_2 = \mathbb{R}^6} \mathscr{W}_*(V_1)^t \otimes \mathscr{W}_*(V_2)^t.$$

Now we study the morphism between the respective hyperhomology spectral sequences $E^r_{*,*}$ and $F^r_{*,*}$ of $\mathbb{H}_*(O(6),\mathscr{W}_*(6)^t)$ and $\mathbb{H}_*(O(6),\bigoplus_{V_1 \oplus^\perp V_2 = \mathbb{R}^6} \mathscr{W}_*(V_1)^t \otimes \mathscr{W}_*(V_2))^t$, induced by this deconcatenation map. To this end, we combine two observations related to the calculations in Example 5.1.4.

First at the E^1 level, the induced map is nothing else than the classical coproduct of the tensor algebra $\mathbf{Ts\bar{A}}$, viewed as a commutative Hopf algebra, where the primitives are given by $\bar{\mathbf{A}}$. This translates into the fact that the induced map $E^1_{1,5} \to F^1_{1,5}$ is the zero map, the map $E^1_{2,4} \to F^1_{2,4}$ is bijective, and the map $E^1_{3,3} \to F^1_{3,3}$ is injective.

Secondly, a further inspection of the spectral sequences, using Corollary 3.2.2, shows that

$$E^\infty_{2,4} = E^2_{2,4} \subset E^1_{2,4}, \quad E^\infty_{3,3} = E^3_{3,3} \subset E^2_{3,3} \subset E^1_{3,3},$$

and

$$F^\infty_{2,4} = F^2_{2,4} \subset F^1_{2,4}, \quad F^\infty_{3,3} = F^3_{3,3} \subset F^2_{3,3} \subset F^1_{3,3}.$$

The theorem is an easy consequence of these two observations.

Remark 10.1.5. For a commutative Hopf algebra, it is known that the primitive part injects into the indecomposable part [13]. The cohomology group $H^1(\mathscr{D}(3))$, isomorphic to $H_{1,5}(\mathbf{A},\mathbb{Q})_6$, is the component for $n=6$ of the primitive part of $\mathbf{H}_0\mathbf{St}$. On the other hand, the indecomposables of $\mathbf{H}_0\mathbf{St}$ are given by $\mathbf{H}_0\mathbf{St}^{(1)} := \oplus_n H_0(O(n),\mathscr{S}t^{(1)}(n)^t)$. Then the indecomposable part for $n=6$ is given by $H_0(O(6),\mathscr{S}t^{(1)}(6)^t)$, which is isomorphic from the calculations of Example 5.1.4 to:

$$H_{1,5}(\mathbf{A},\mathbb{Q})_6 \oplus H_2^{(1)}(\mathbf{A},\mathbb{Q})_6 \oplus \mathrm{Ker}\{d^2 : H_{3,3}^{(1)}(\mathbf{A},\mathbb{Q})_6 \to H_{1,2}(\mathbf{A},\mathbb{Q})_6\}.$$

This measures the gap between the primitives and the indecomposables.

10.2. Dehn complexes for hyperbolic and Euclidean polytopes

The m-th Dehn complexes here are the $2m$-graded part of the cobar complex of the Sah algebra with coefficients in scissors congruence groups of hyperbolic or Euclidean polytopes and look like

$$\begin{aligned}\mathscr{D}^h(m) : \mathscr{P}(H^{2m-1}) &\to \bigoplus_{m_1+m_2=m} \mathscr{P}(H^{2m_1-1}) \otimes \bar{\mathscr{P}}(S^{2m_2-1}) \to \cdots\\ &\to \bigoplus_{m_1+\cdots+m_s=m} \mathscr{P}(H^{2m_1-1}) \otimes \bar{\mathscr{P}}(S^{2m_2-1}) \otimes \cdots \otimes \bar{\mathscr{P}}(S^{2m_s-1})\\ &\to \cdots \to \mathscr{P}(H^1) \otimes \left(\overset{m-1}{\otimes} \bar{\mathscr{P}}(S^1)\right),\end{aligned}$$

and

$$\begin{aligned}\mathscr{D}^e(m) : \mathscr{P}(E^{2m-1}) &\to \bigoplus_{m_1+m_2=m} \mathscr{P}(E^{2m_1-1}) \otimes \bar{\mathscr{P}}(S^{2m_2-1}) \to \cdots\\ &\to \bigoplus_{m_1+\cdots+m_s=m} \mathscr{P}(E^{2m_1-1}) \otimes \bar{\mathscr{P}}(S^{2m_2-1}) \otimes \cdots \otimes \bar{\mathscr{P}}(S^{2m_s-1})\\ &\to \cdots \to \mathscr{P}(E^1) \otimes \left(\overset{m-1}{\otimes} \bar{\mathscr{P}}(S^1)\right),\end{aligned}$$

where $\mathscr{P}(H^{2m-1})$ and $\mathscr{P}(E^{2m-1})$ are placed in cohomological degree one (we adopt the convention of Goncharov: this convention is justified when passing to the complexified situation. We plan to consider this aspect in a sequel to this article).

Theorem 10.2.1. *The cohomology of the Dehn complexes $\mathscr{D}^\star(m)$, $\star=h,e$, in degrees m and $m-1$, is respectively given by*

$$H^m(\mathscr{D}^h(m)) \cong H_m(O(1,2m-1),\mathbb{Q}^t),$$
$$H^{m-1}(\mathscr{D}^h(m)) \cong H_{m+1}(O(1,2m-1),\mathbb{Q}^t),$$

and

$$H^m(\mathscr{D}^e(m)) \cong H_m(E(1,2m-1),\mathbb{Q}^t),$$
$$H^{m-1}(\mathscr{D}^e(m)) \cong H_{m+1}(E(1,2m-1),\mathbb{Q}^t).$$

The proof is a consequence of the isomorphisms (20) and (21) of Section 9.4. This is analogous to the proof of Theorem 10.1.1. The particular case of dimension 3 ($m=2$) was a result of Dupont.

Analogously to Proposition 10.1.2, we have:

Proposition 10.2.2. *The natural maps*

$$H_1(O(1,1),\mathbb{Q}^t)\otimes(\otimes^{m-1}H_1(O(2),\mathbb{Q}^t))\to H_m(O(1,2m-1),\mathbb{Q}^t),$$
$$H_1(E(1,1),\mathbb{Q}^t)\otimes(\otimes^{m-1}H_1(O(2),\mathbb{Q}^t))\to H_m(E(1,2m-1),\mathbb{Q}^t),$$

are surjective.

For the hyperbolic and Euclidean spaces in dimension 5, we can precise the cohomology groups of Dehn complexes as follows:

Theorem 10.2.3. *The cohomology groups of the Dehn complexes $\mathscr{D}^h(3)$ and $\mathscr{D}^e(3)$ read*

$$H^1(\mathscr{D}^h(3))\cong H_{0,5}(\mathbf{A},\mathbf{M}^h)_6,\ \ H^2(\mathscr{D}^h(3))\cong H_4(O(1,5),\mathbb{Q}^t),$$
$$H^3(\mathscr{D}^h(3))\cong H_3(O(1,5),\mathbb{Q}^t),$$

and

$$H^1(\mathscr{D}^e(3))\cong H_{0,5}(\mathbf{A},\mathbf{M}^e)_6,\ \ H^2(\mathscr{D}^e(3))\cong H_4(E(1,5),\mathbb{Q}^t),$$
$$H^3(\mathscr{D}^e(3))\cong H_3(E(1,5),\mathbb{Q}^t).$$

The proof is similar to the one of Theorem 10.1.4.

The isomorphism $H^1(\mathscr{D}^e(3))\cong H_{0,5}(\mathbf{A},\mathbf{M}^e)_6$ was implicit in Dupont [2], and actually his computation was at the origin of the present work.

The groups $H^1(\mathscr{D}^h(3))$ and $H^1(\mathscr{D}^e(3))$ appear respectively as quotients of the homology groups $H_5(O(1,5),\mathbb{Q}^t)$ and $H_5(E(1,5),\mathbb{Q}^t)$.

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